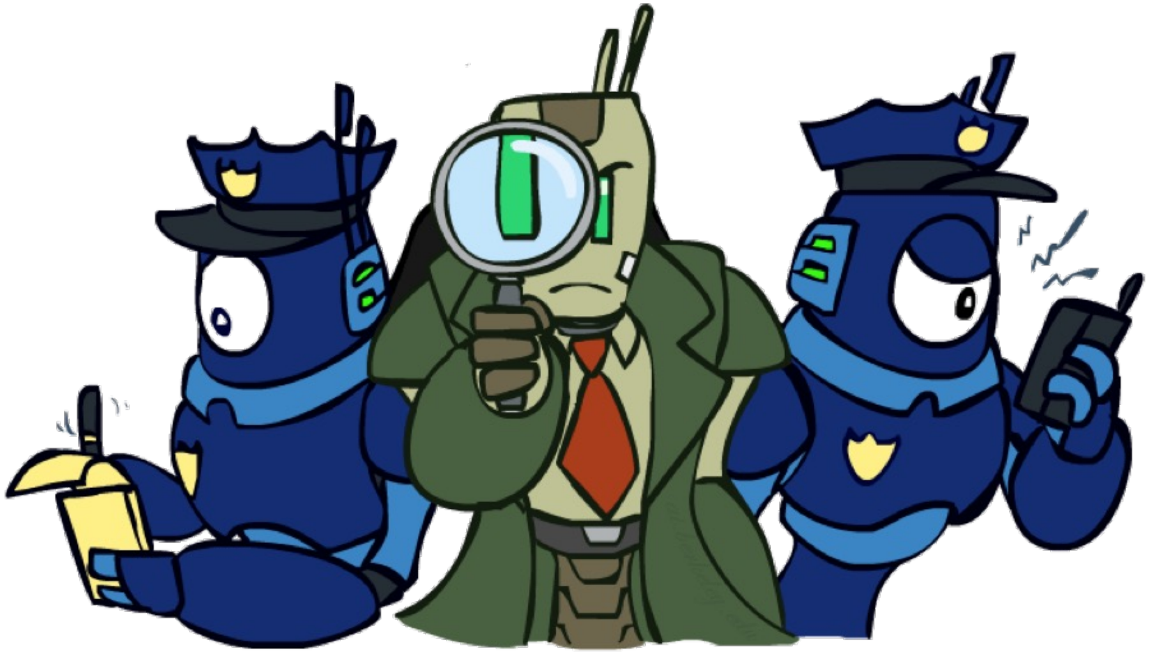
CMSC 471

Constraint Satisfaction Problems III



Instructor: KMA Solaiman

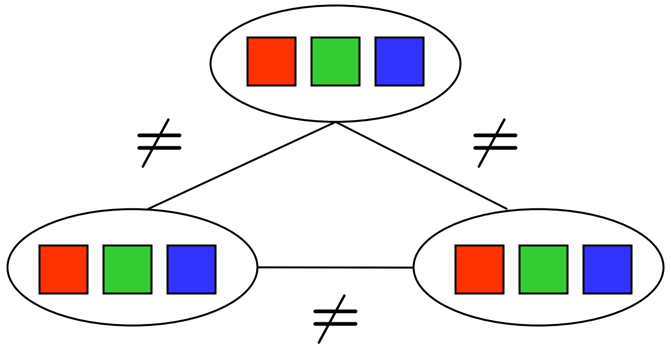
These slides were modified from Dan Klein and Pieter Abbeel at UC Berkeley [ai.berkeley.edu].

Today

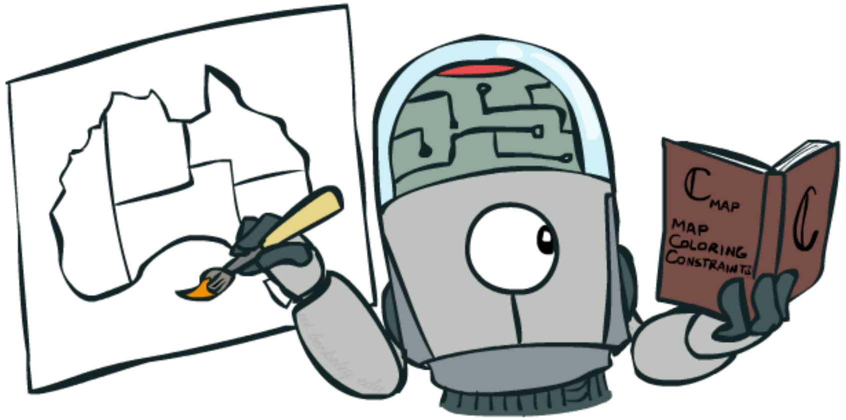
- Efficient Solution of CSPs
- Local Search

Reminder: CSPs

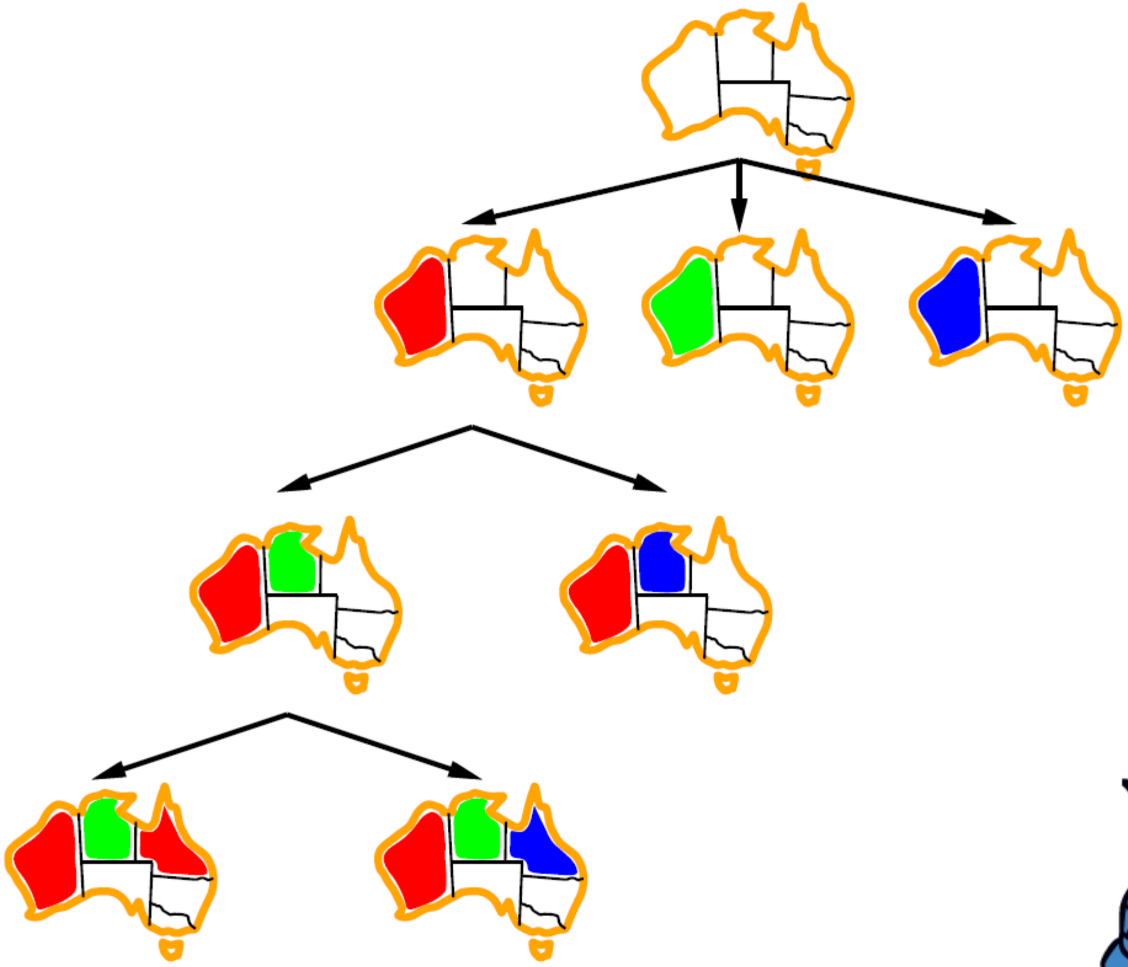
- CSPs:
  - Variables
  - Domains
  - Constraints
    - Implicit (provide code to compute)
    - Explicit (provide a list of the legal tuples)
    - Unary / Binary / N-ary

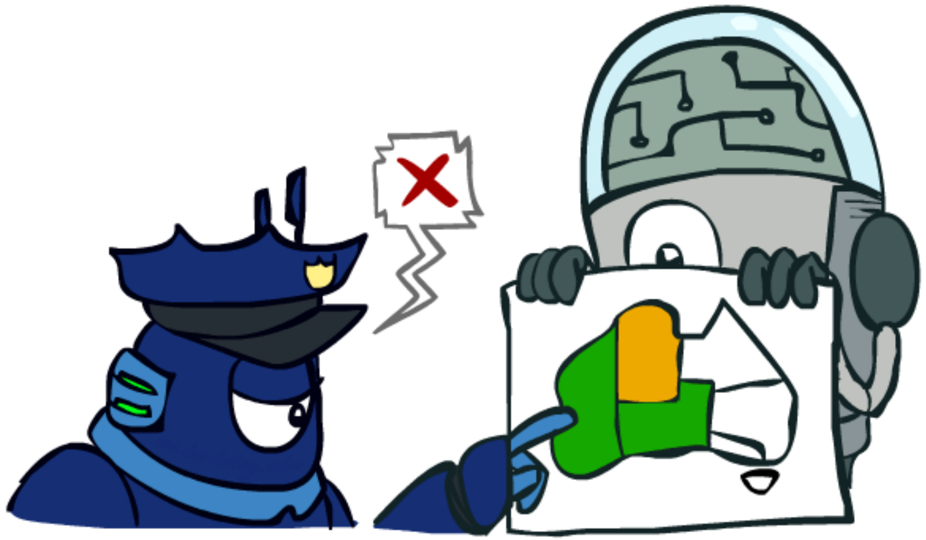


- Goals:
  - Here: find any solution
  - Also: find all, find best, etc.



Backtracking Example





Improving Backtracking

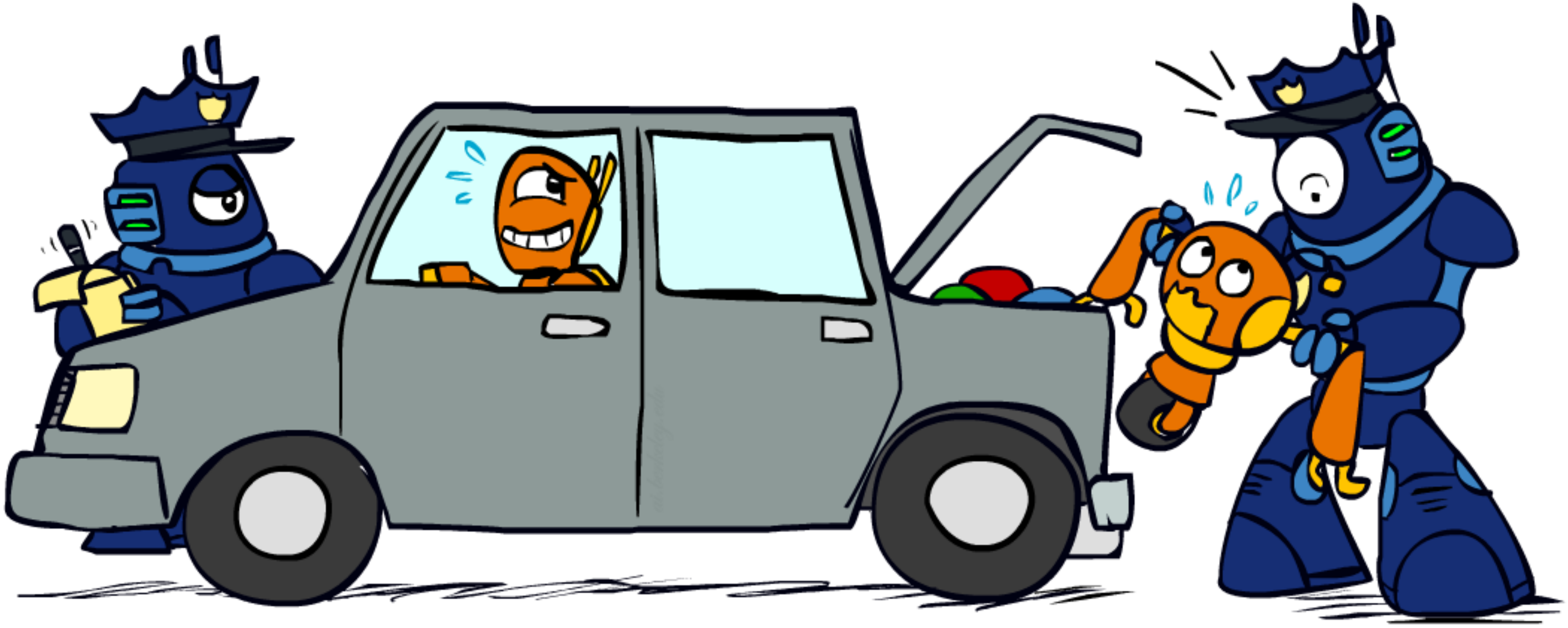
- General-purpose ideas give huge gains in speed
  - … but it's all still NP-hard

- Filtering: Can we detect inevitable failure early?

- Ordering:
  - Which variable should be assigned next? (MRV)
  - In what order should its values be tried? (LCV)

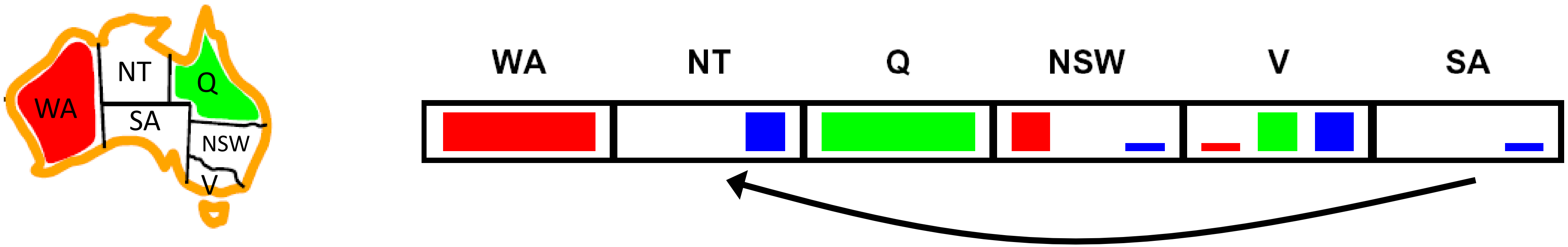
- Structure: Can we exploit the problem structure?

Arc Consistency and Beyond



Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are consistent:



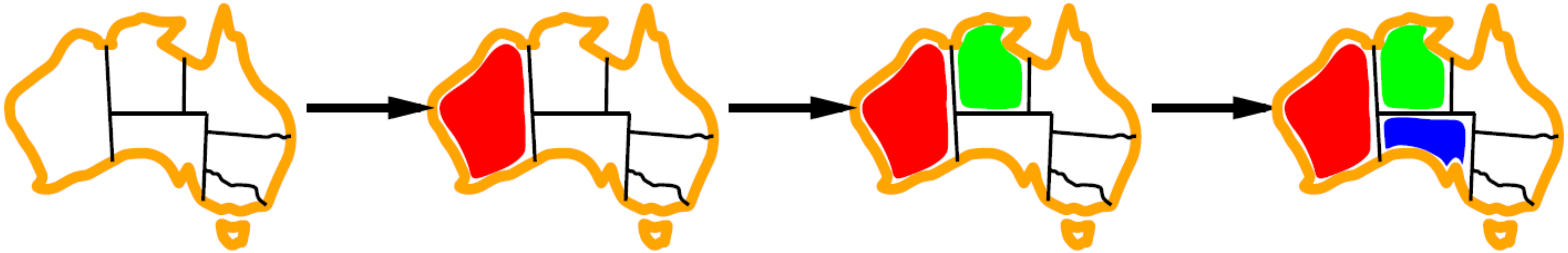
- Important: If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What's the downside of enforcing arc consistency?

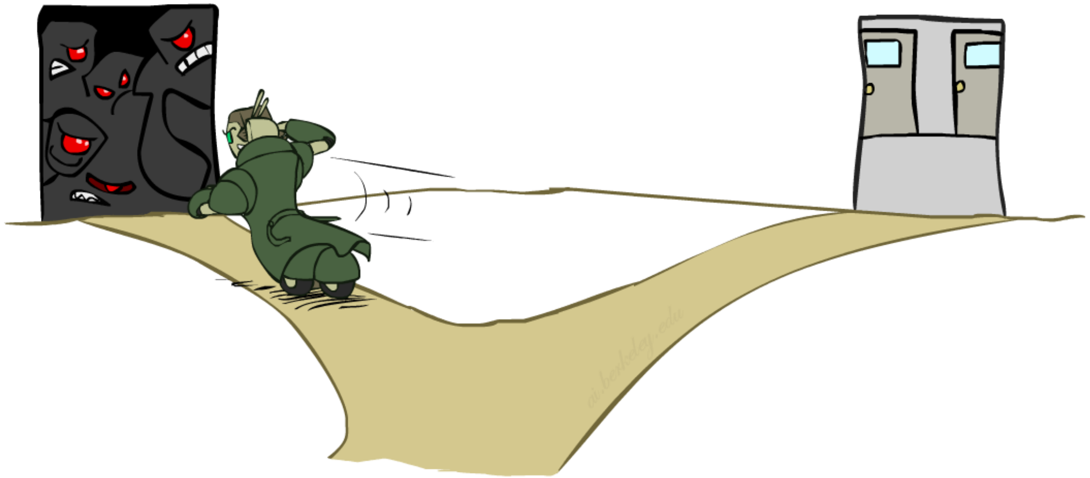
Remember: Delete from the tail!

Ordering

Ordering: Minimum Remaining Values

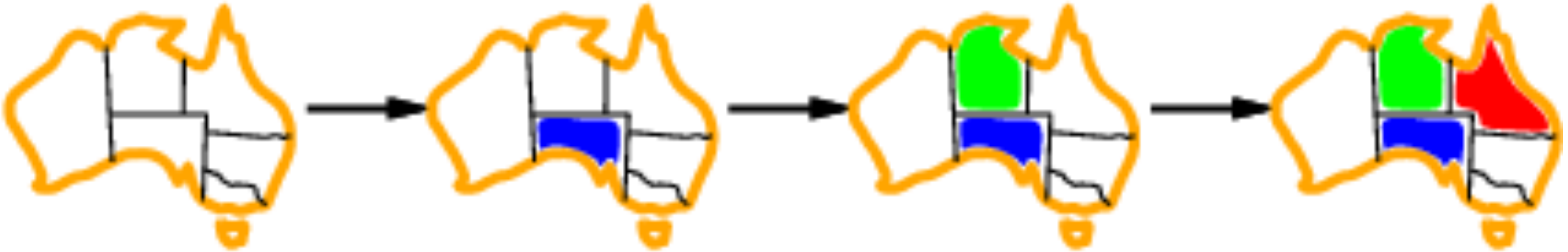
- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain
  - Aka most constrained variables





- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering

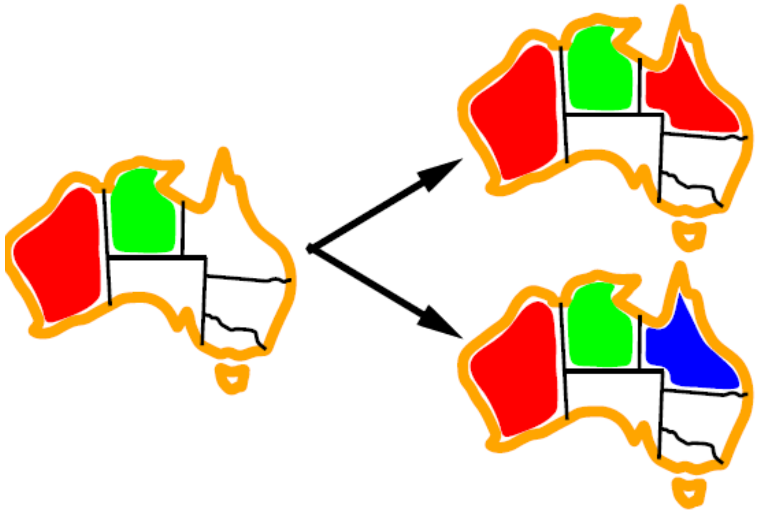
- Tie-breaker among Minimum remaining values
- Choose variable involved in largest # of constraints on remaining variables



- After assigning SA to be blue, WA, NT, Q, NSW and V all have just two values left.
- But WA and V have only one constraint (WA has constraint with NT, and V with NSW) on remaining variables and T none, so choose one of NT, Q & NSW (each of which has 2 cons. left)

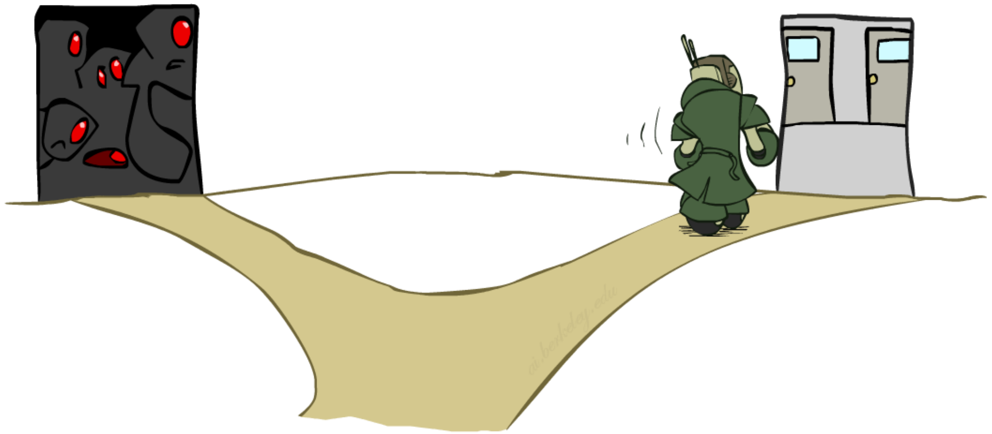
Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
  - I.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this! (E.g., rerunning filtering)

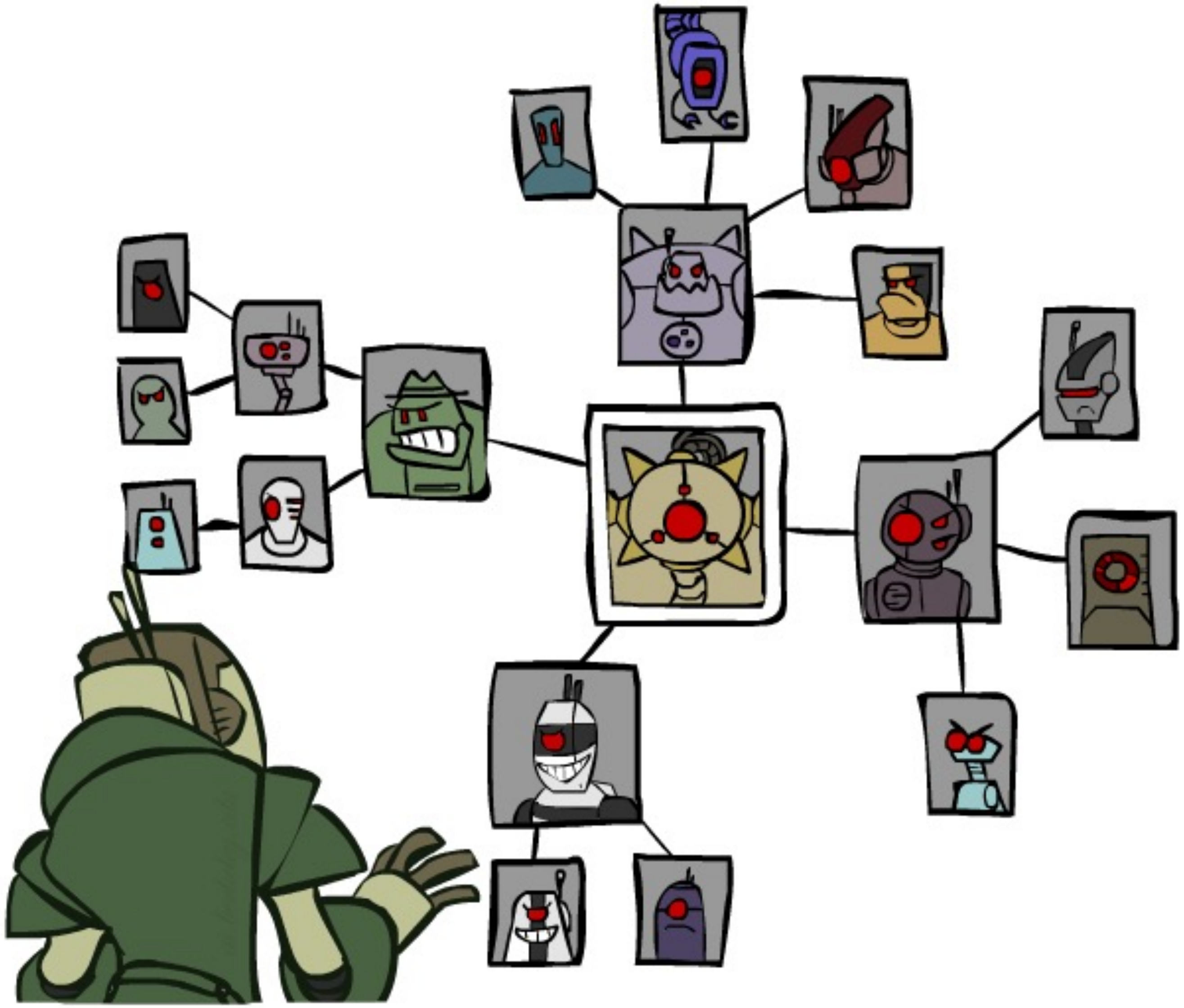


- Why least rather than most?

- Combining these ordering ideas makes 1000 queens feasible

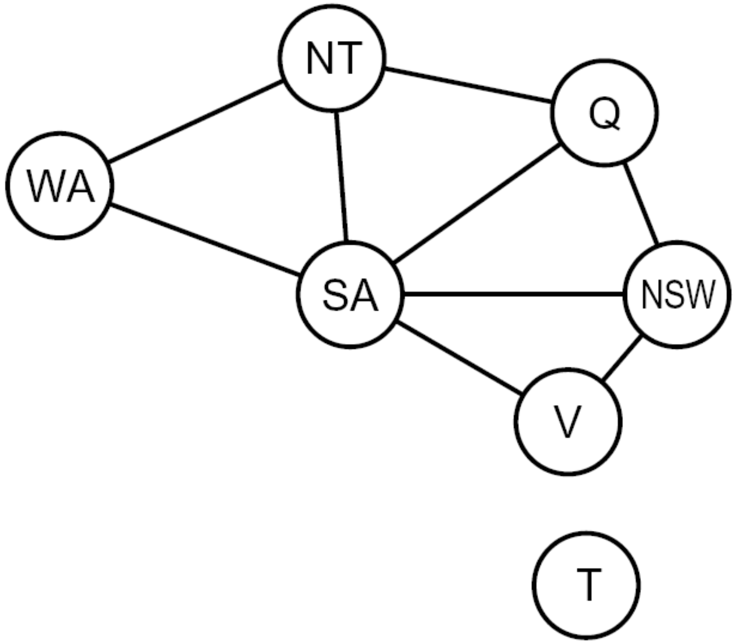


Structure

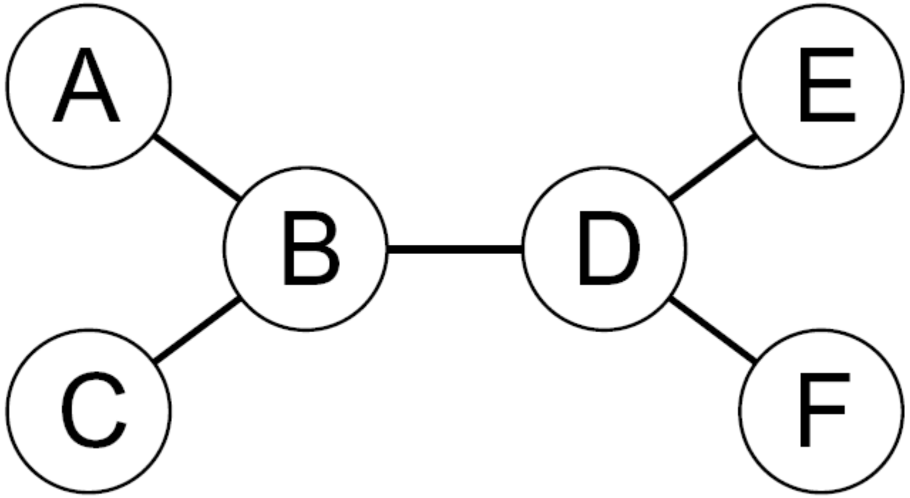


Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact
- Independent subproblems are identifiable as connected components of constraint graph
- Suppose a graph of n variables can be broken into subproblems of only c variables:
  - Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c =20
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec



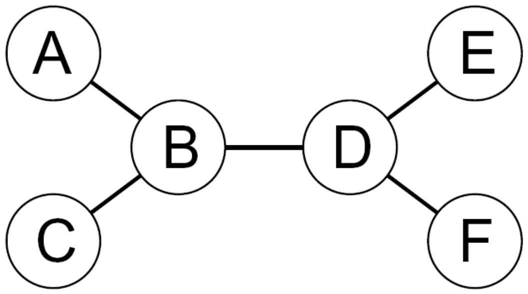
Tree-Structured CSPs



- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\ d^2)$ time
  - Compare to general CSPs, where worst-case time is $O(d^n)$
- This property also applies to probabilistic reasoning (later): an example of the relation between syntactic restrictions and the complexity of reasoning

Tree-Structured CSPs

- Algorithm for tree-structured CSPs:
  - Order: Choose a root variable, order variables so that parents precede children



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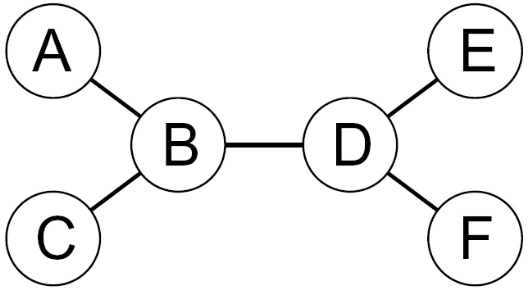

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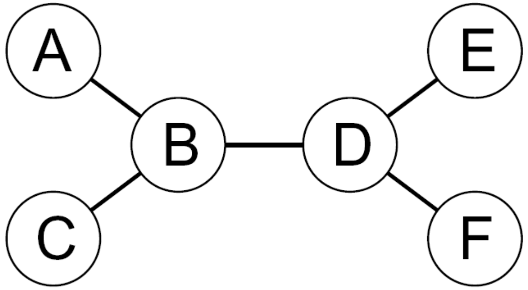

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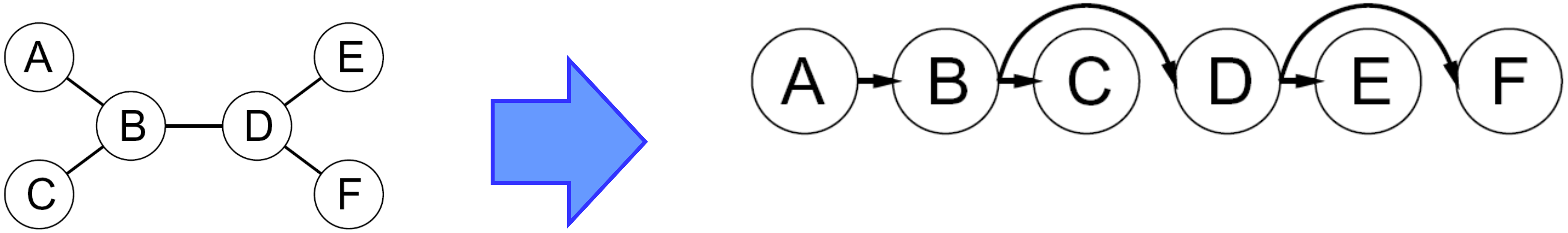

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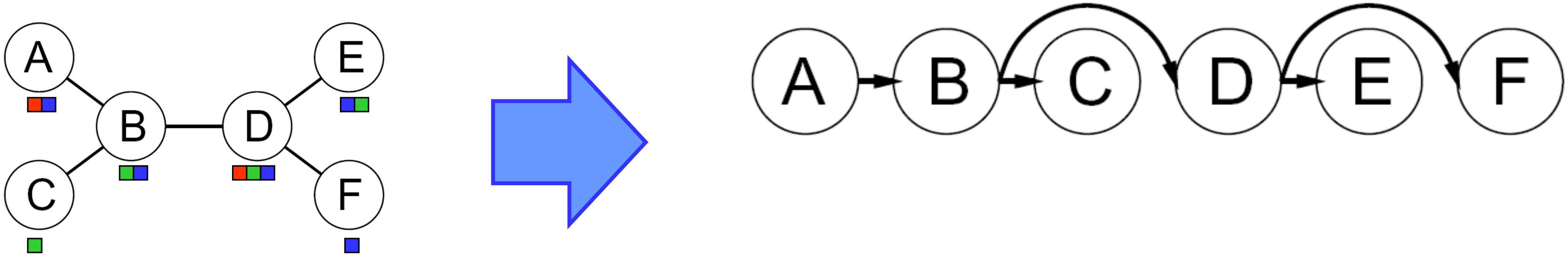

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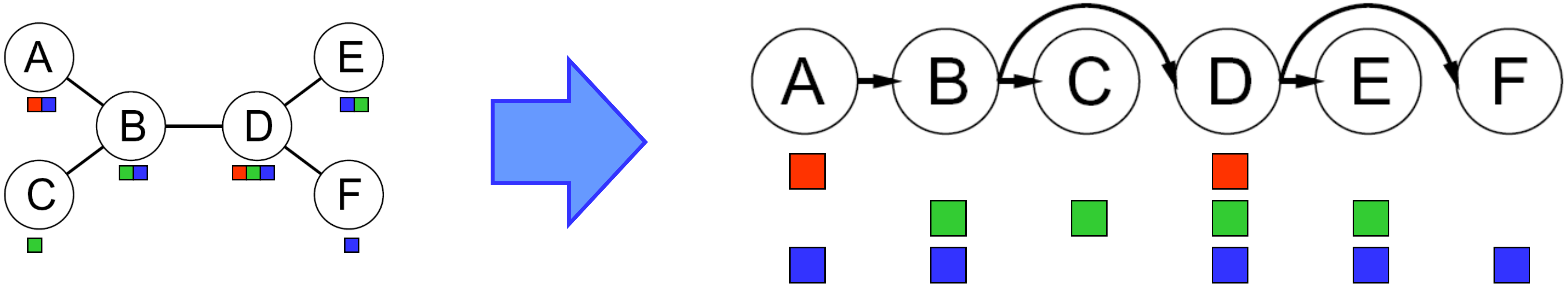

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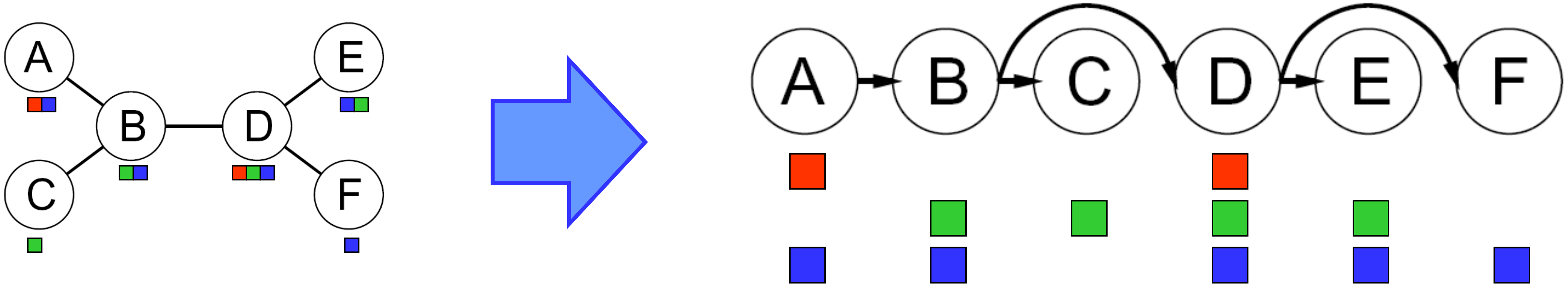

  - Remove backward: For i = n : 2, apply RemoveInconsistent(Parent(X_i),X_i)

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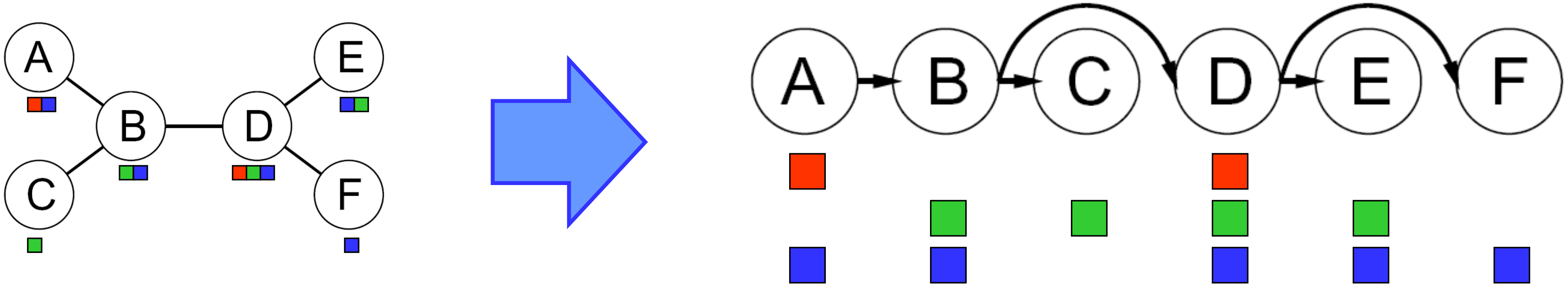

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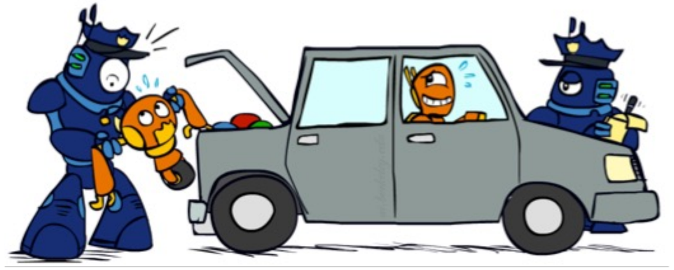

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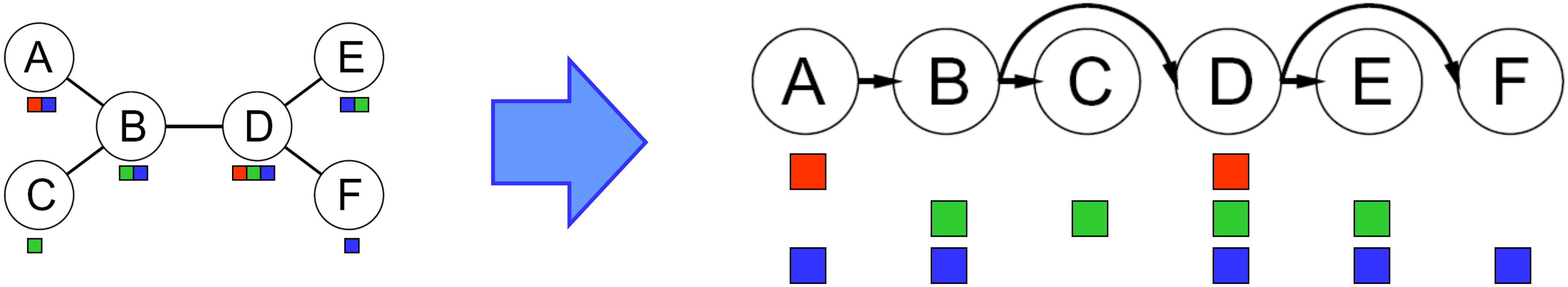

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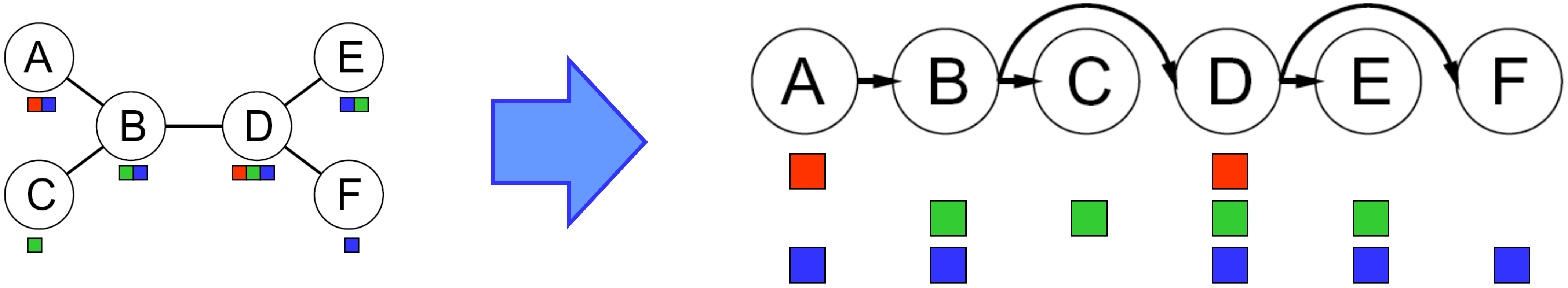

  - Remove backward: For i = n : 2, apply RemoveInconsistent(Parent(X_i),X_i)
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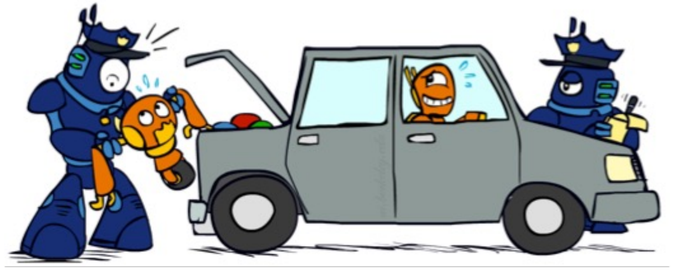
- Runtime: O(n d^2) (why?)



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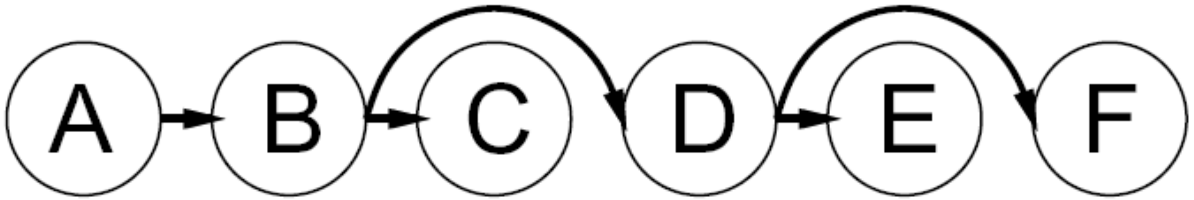
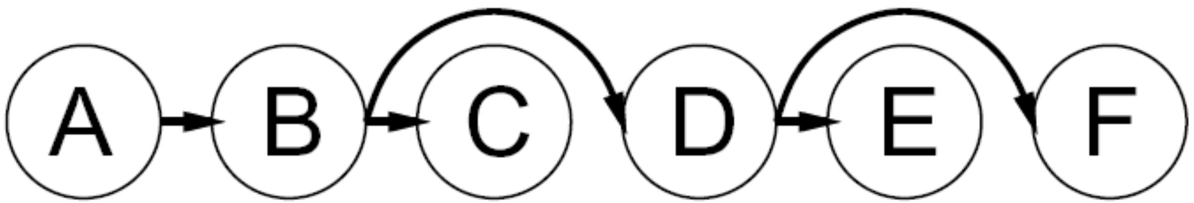


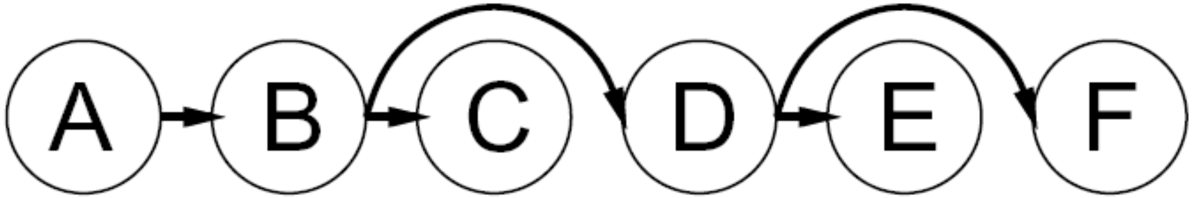
Tree-Structured CSPs

- Claim 1: After backward pass, all root-to-leaf arcs are consistent



Tree-Structured CSPs

- Claim 1: After backward pass, all root-to-leaf arcs are consistent
- Proof: Each X→Y was made consistent at one point and Y's domain could not have been reduced thereafter (because Y's children were processed before Y)



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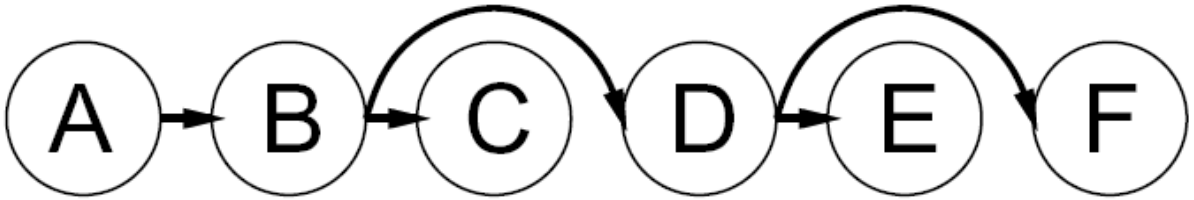

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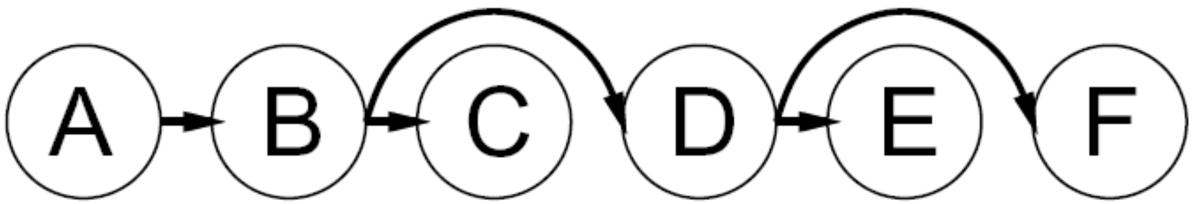

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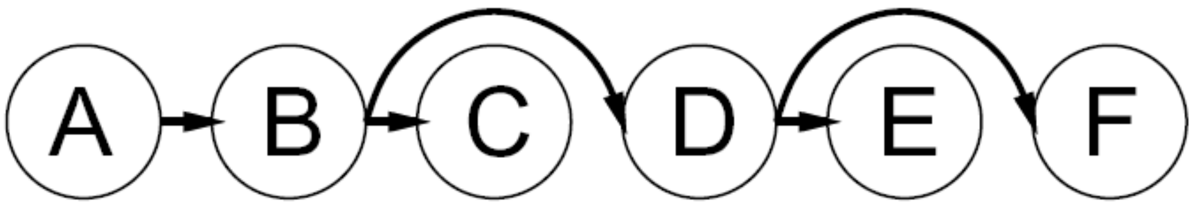

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- Why doesn't this algorithm work with cycles in the constraint graph?

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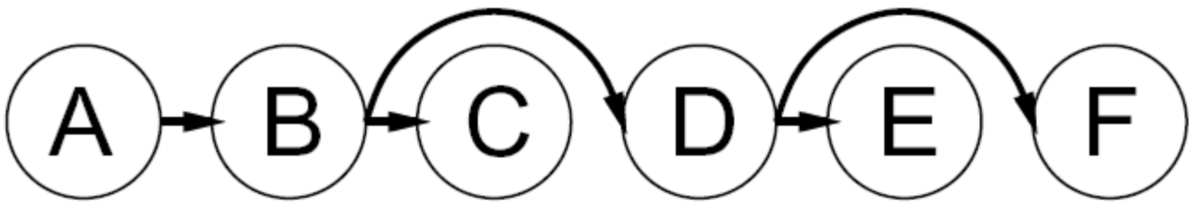

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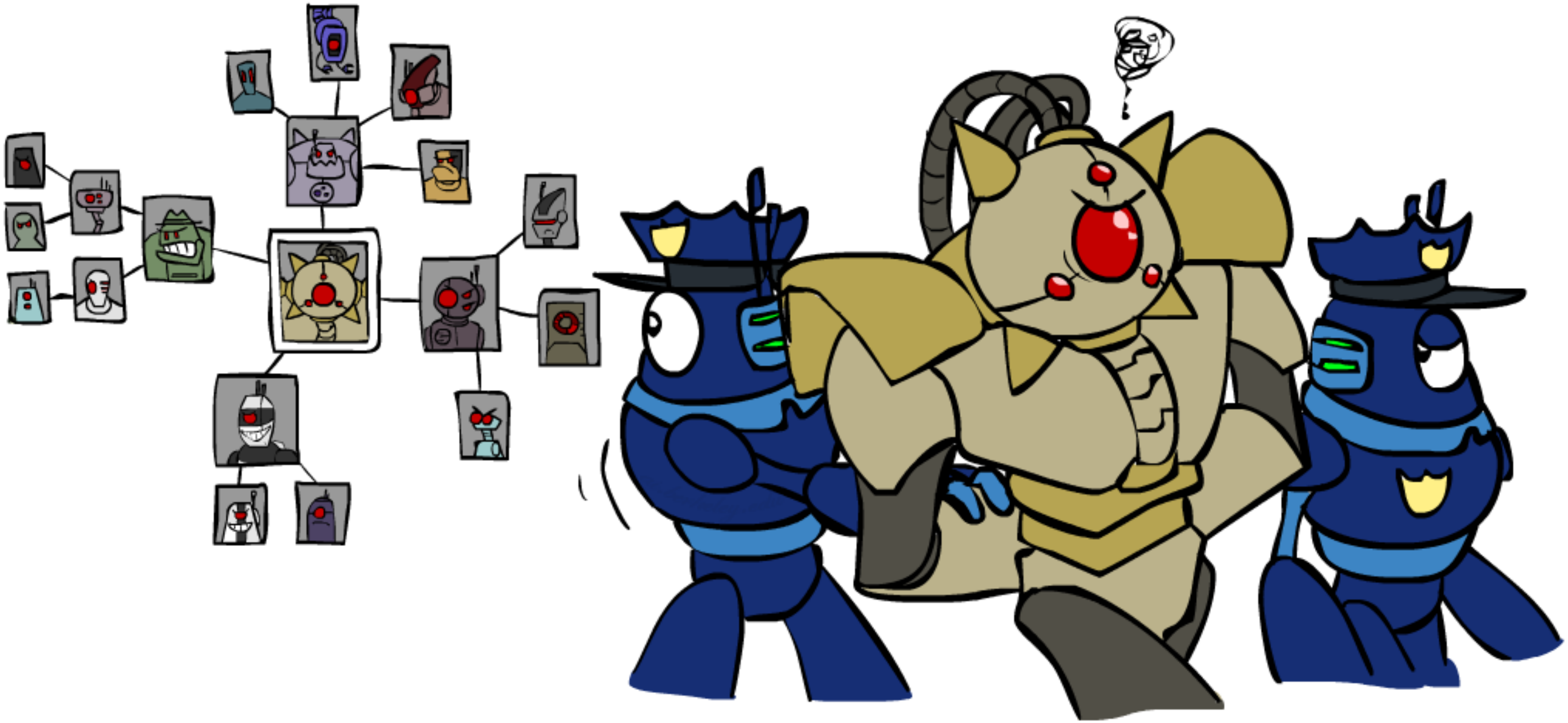
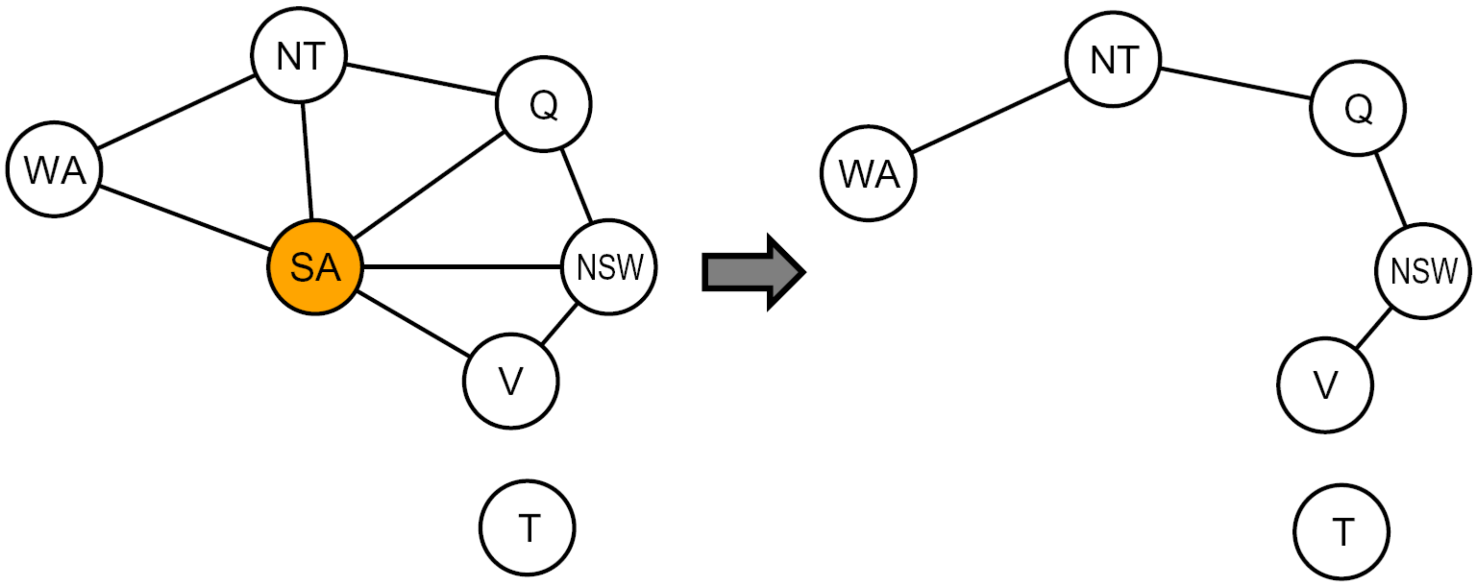
- Note: we'll see this basic idea again with Bayes' nets

Improving Structure



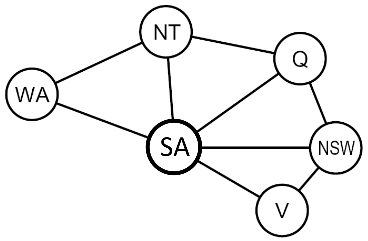
Nearly Tree-Structured CSPs



- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime $O((d^c) (n-c) d^2)$, very fast for small c

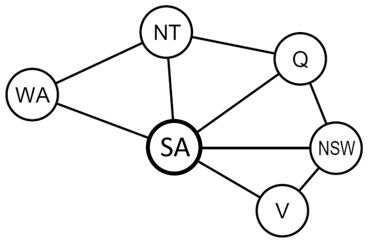
Cutset Conditioning

Cutset Conditioning



Cutset Conditioning

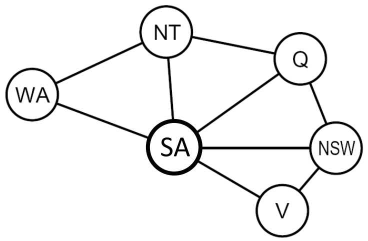
Choose a cutset



Cutset Conditioning

Choose a cutset

Instantiate the cutset
(all possible ways)



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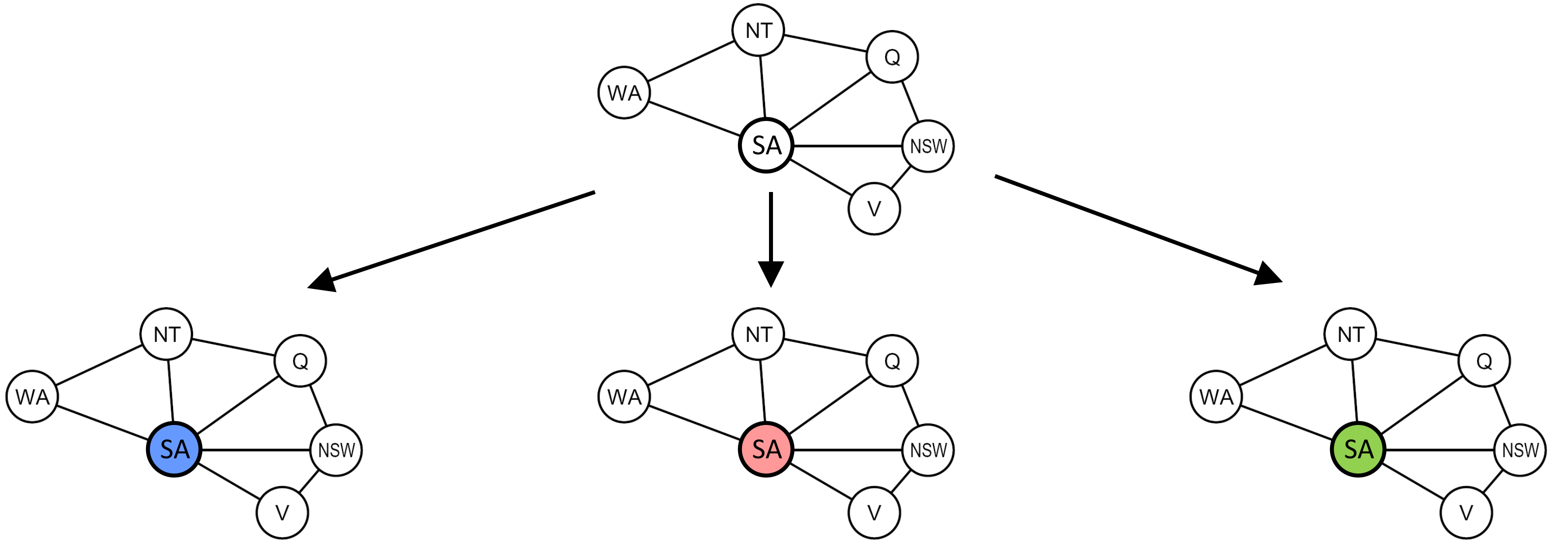

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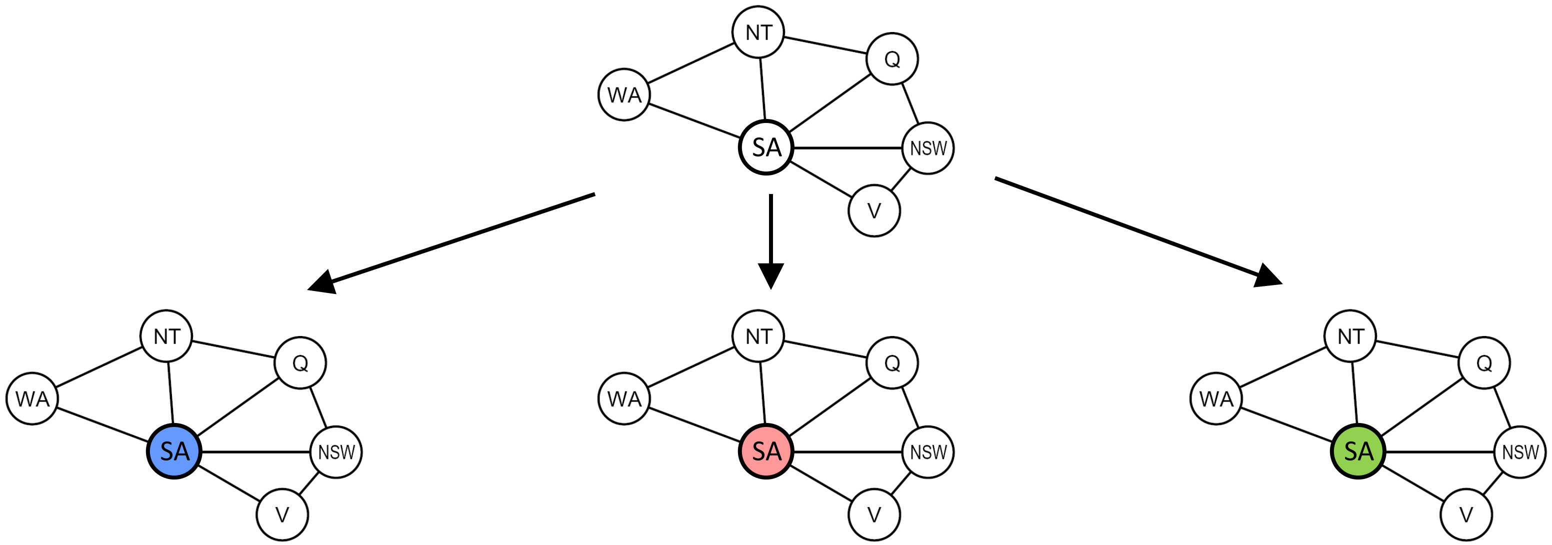
Compute residual CSP for each assignment



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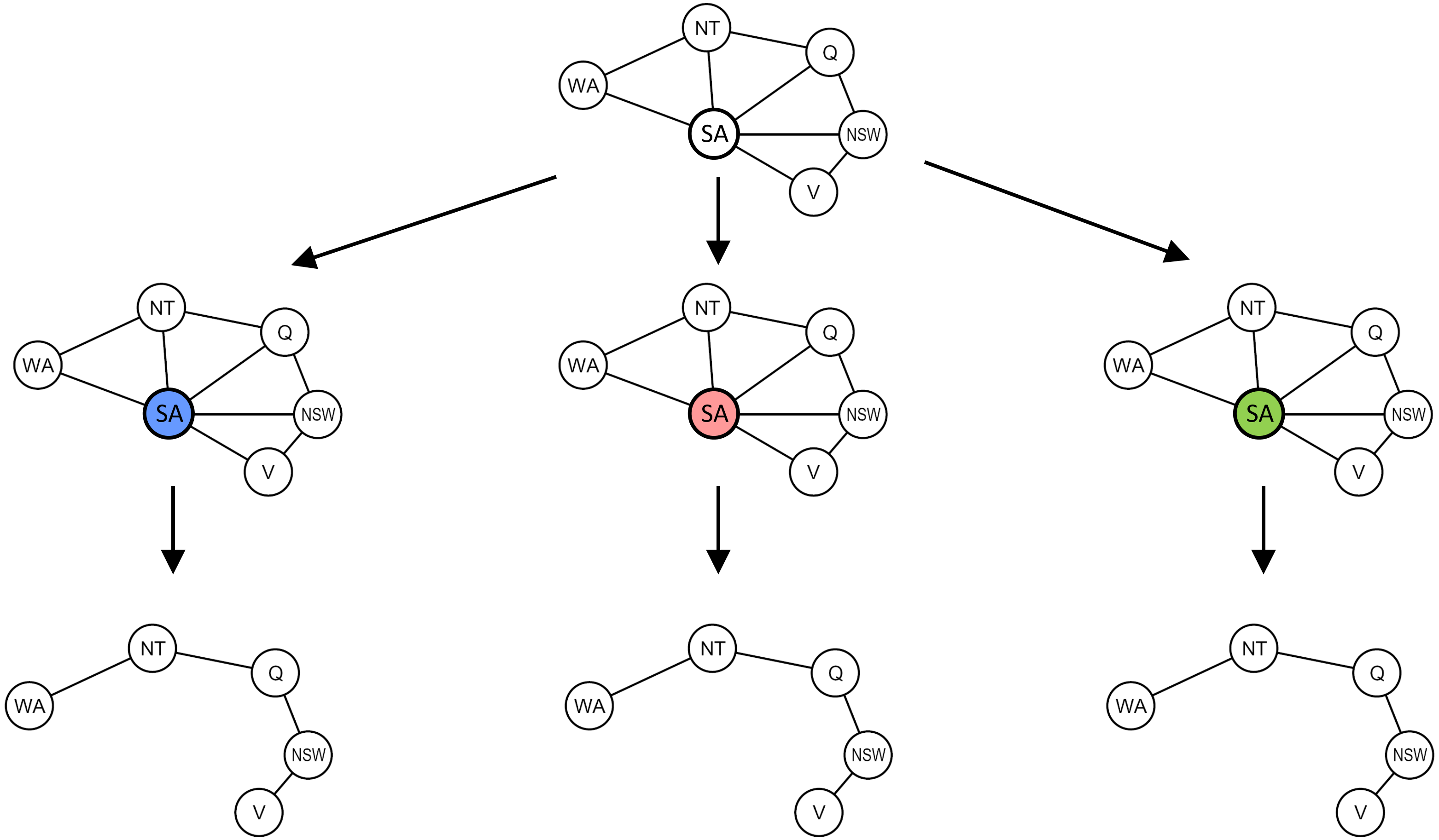

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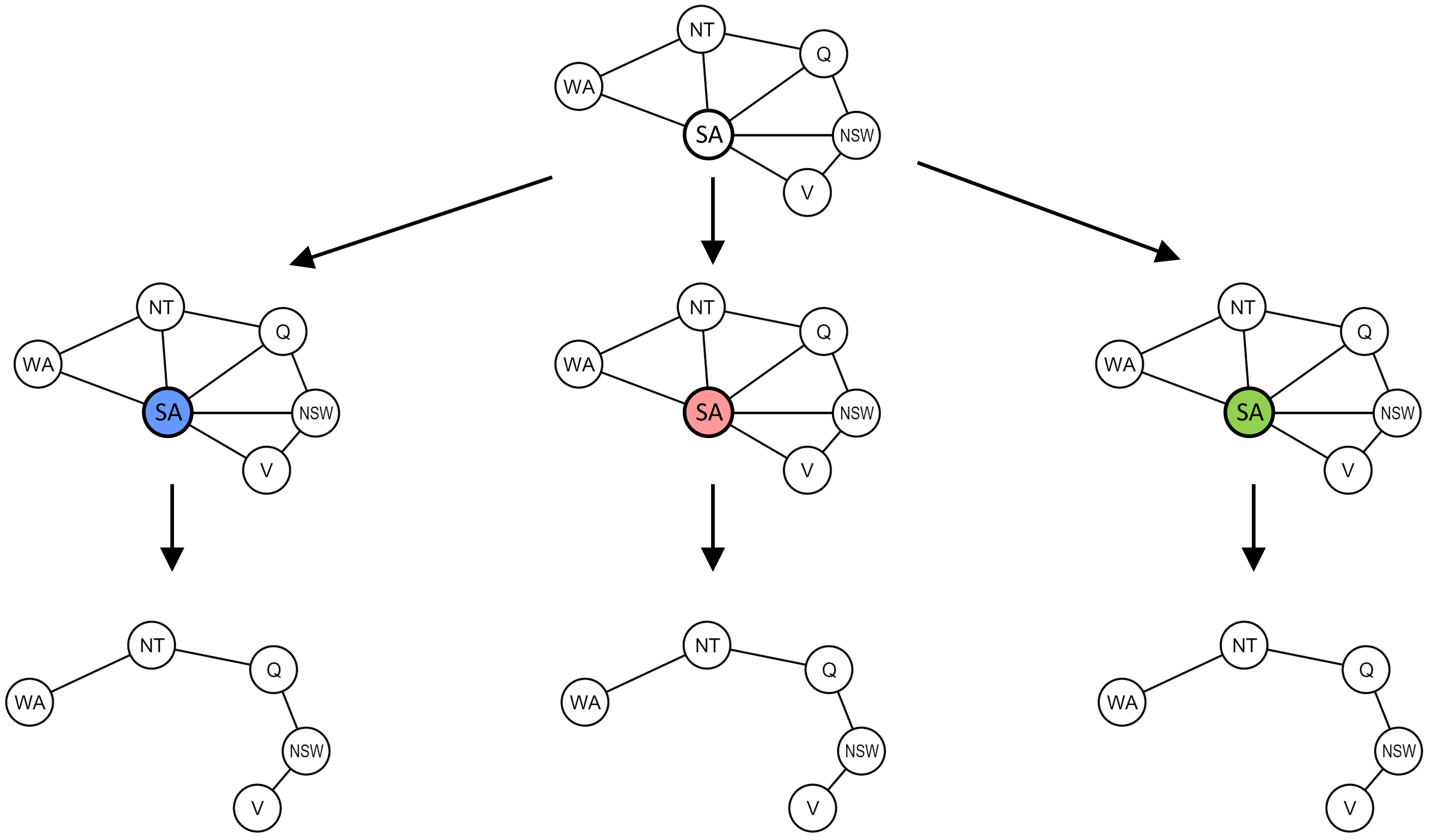
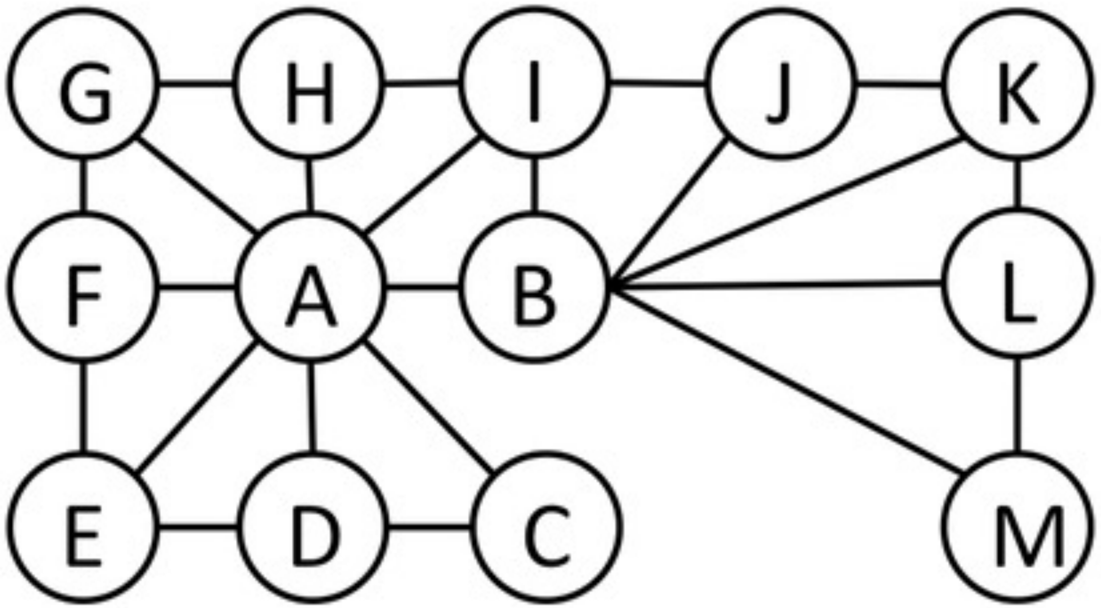
Solve the residual CSPs (tree structured)

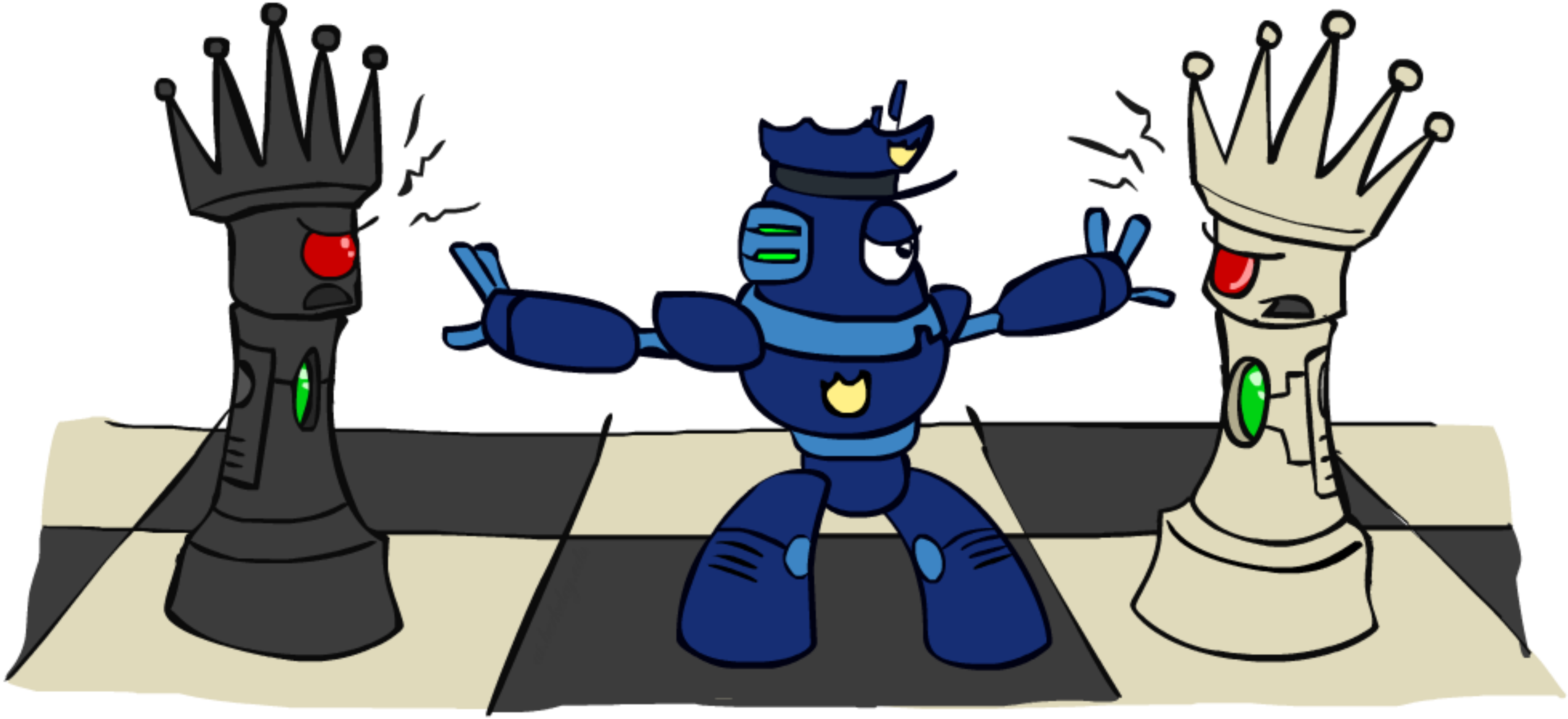


Cutset Quiz

- Find the smallest cutset for the graph below.



Iterative Improvement



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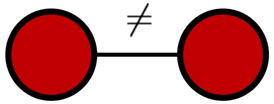

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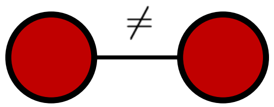
- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe! Live on the edge.



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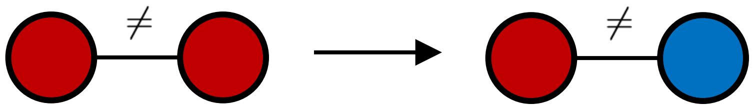

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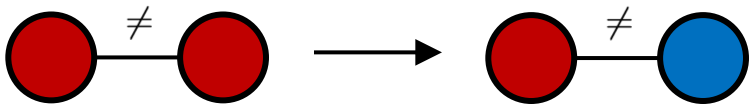
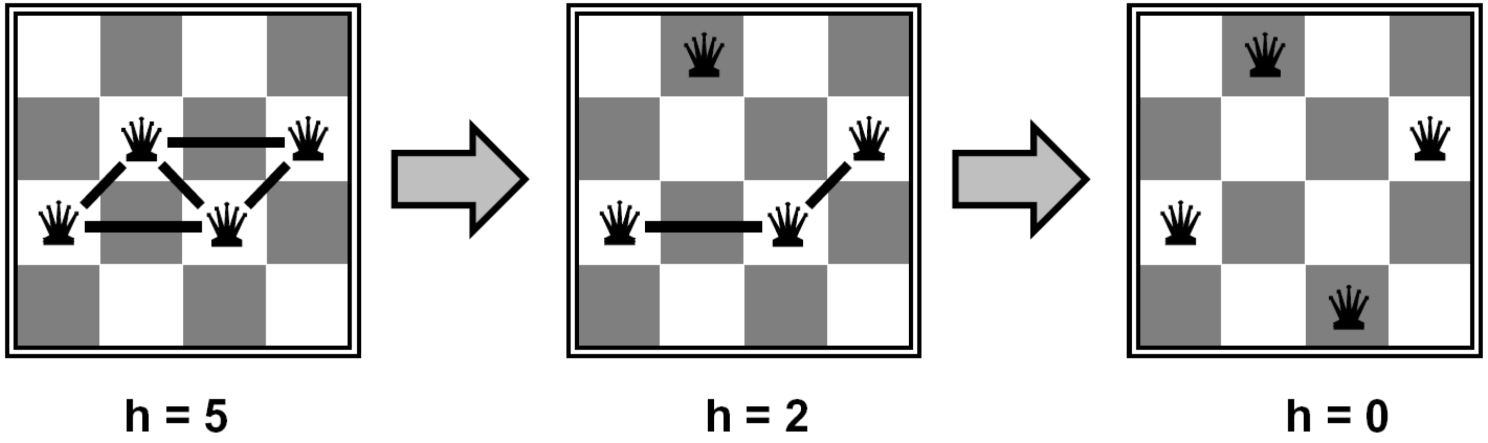


- Algorithm: While not solved,
  - Variable selection: randomly select any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - I.e., hill climb with h(n) = total number of violated constraints

Example: 4-Queens



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- Operators: move queen in column
- Goal test: no attacks
- Evaluation: c(n) = number of attacks

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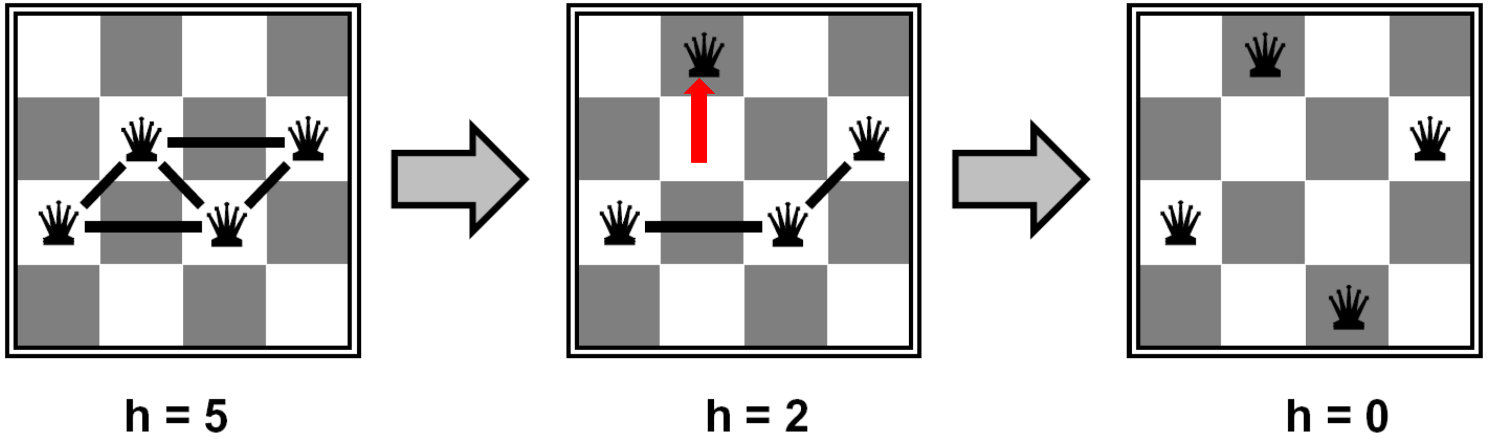

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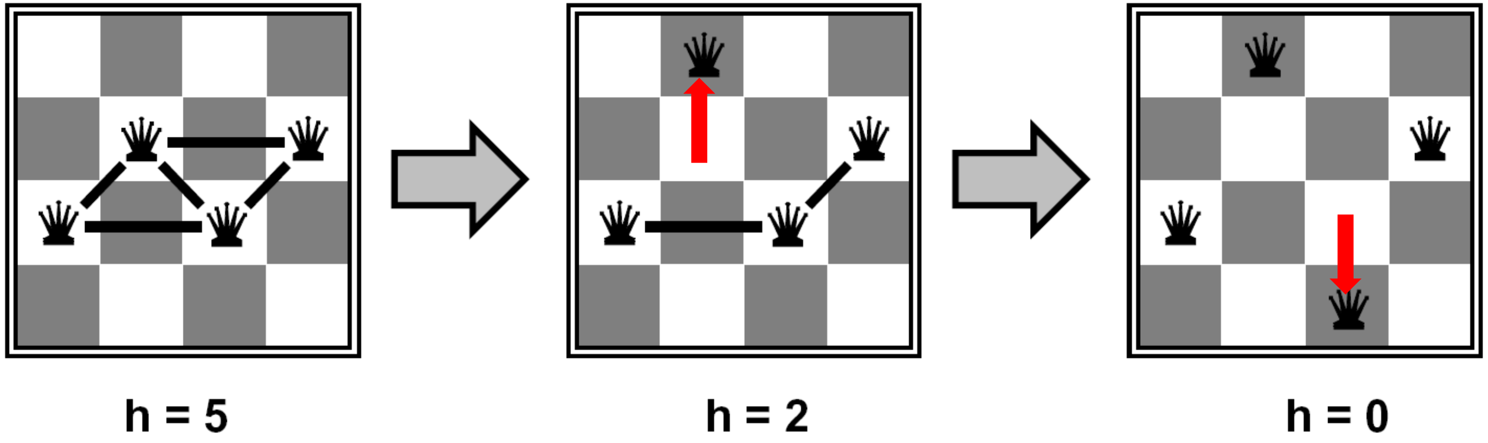


- States: 4 queens in 4 columns (4^4 = 256 states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: c(n) = number of attacks

Basic Local Search Algorithm

Assign one domain value d_i to each variable v_i

while no solution & not stuck & not timed out:

    bestCost $\leftarrow \infty$;   bestList $\leftarrow$ [ ];

    for each variable v_i where Cost(Value(v_i)) > 0

        for each domain value d_i of v_i

            if Cost(d_i) < bestCost

                bestCost $\leftarrow$ Cost(d_i)

                bestList $\leftarrow$ [d_i]

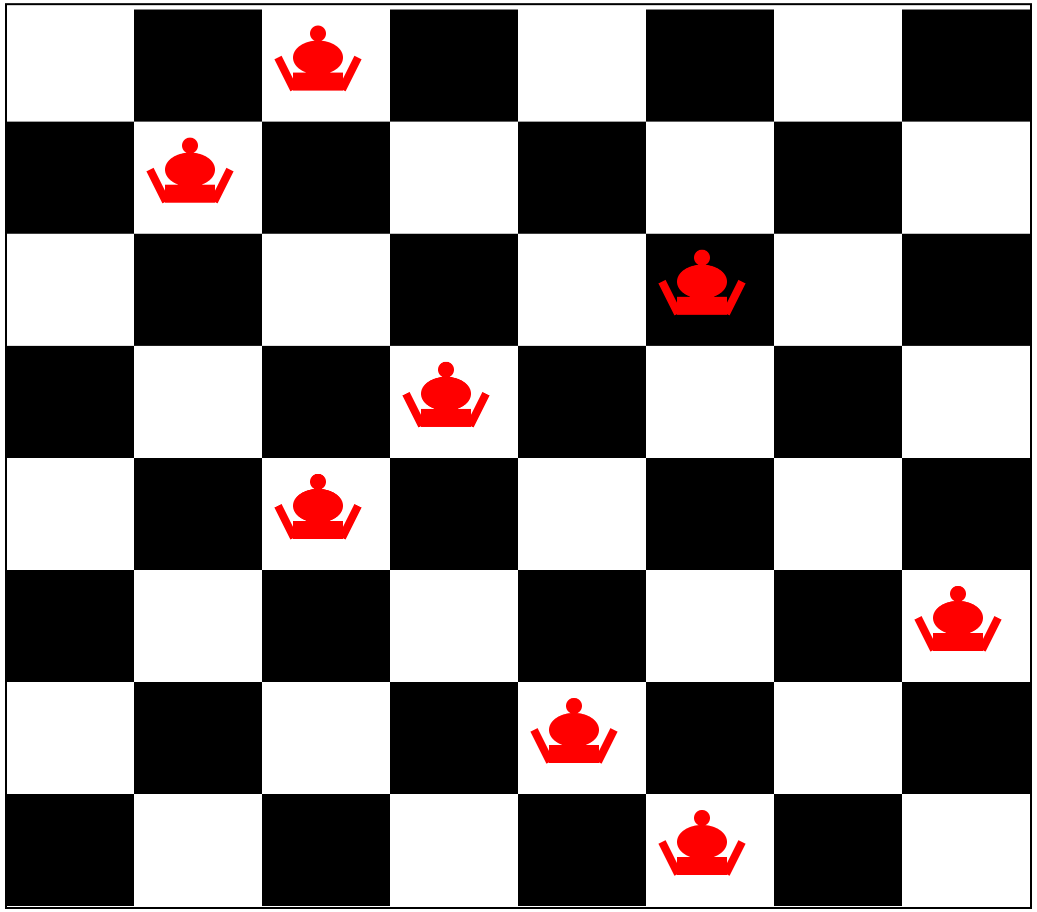
            else if Cost(d_i) = bestCost

                bestList $\leftarrow$ bestList $\cup$ d_i

    Take a randomly selected move from bestList

Eight Queens using Local Search

Place 8 Queens randomly on the board

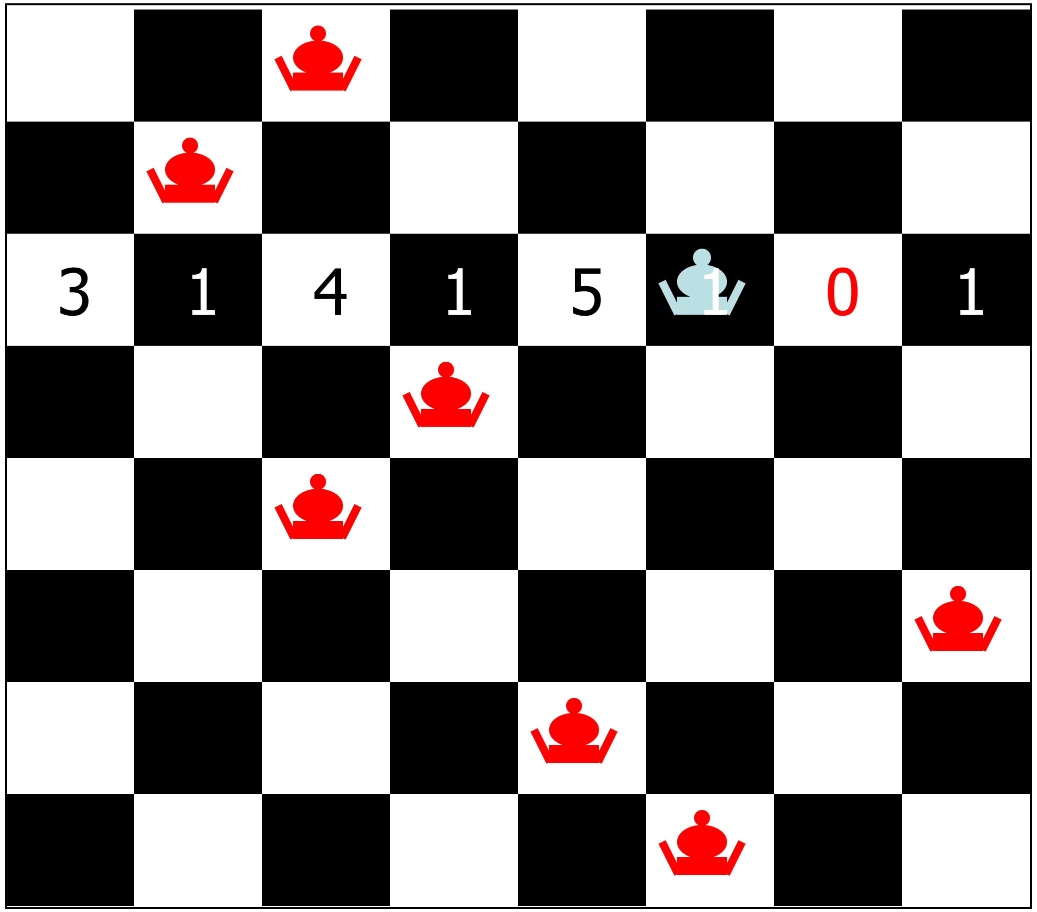


Slide

Eight Queens using Local Search

Pick a Queen:

Calculate cost of each move



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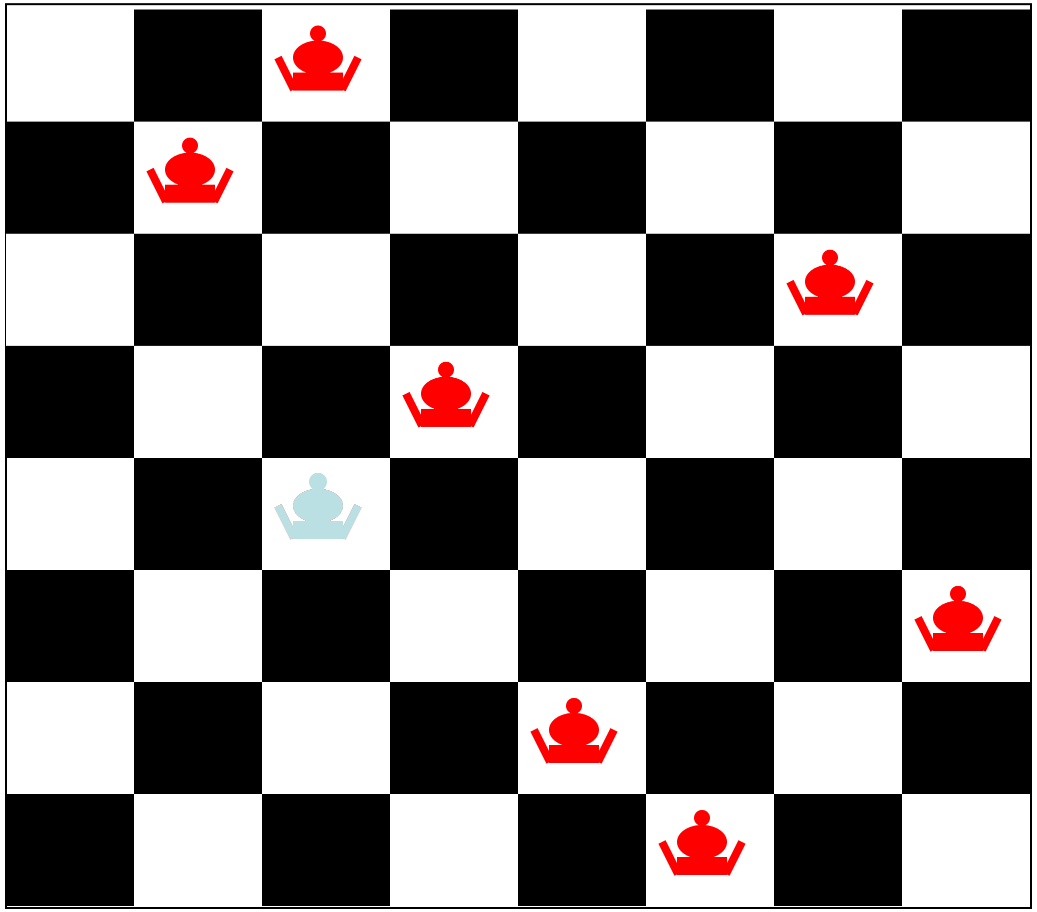
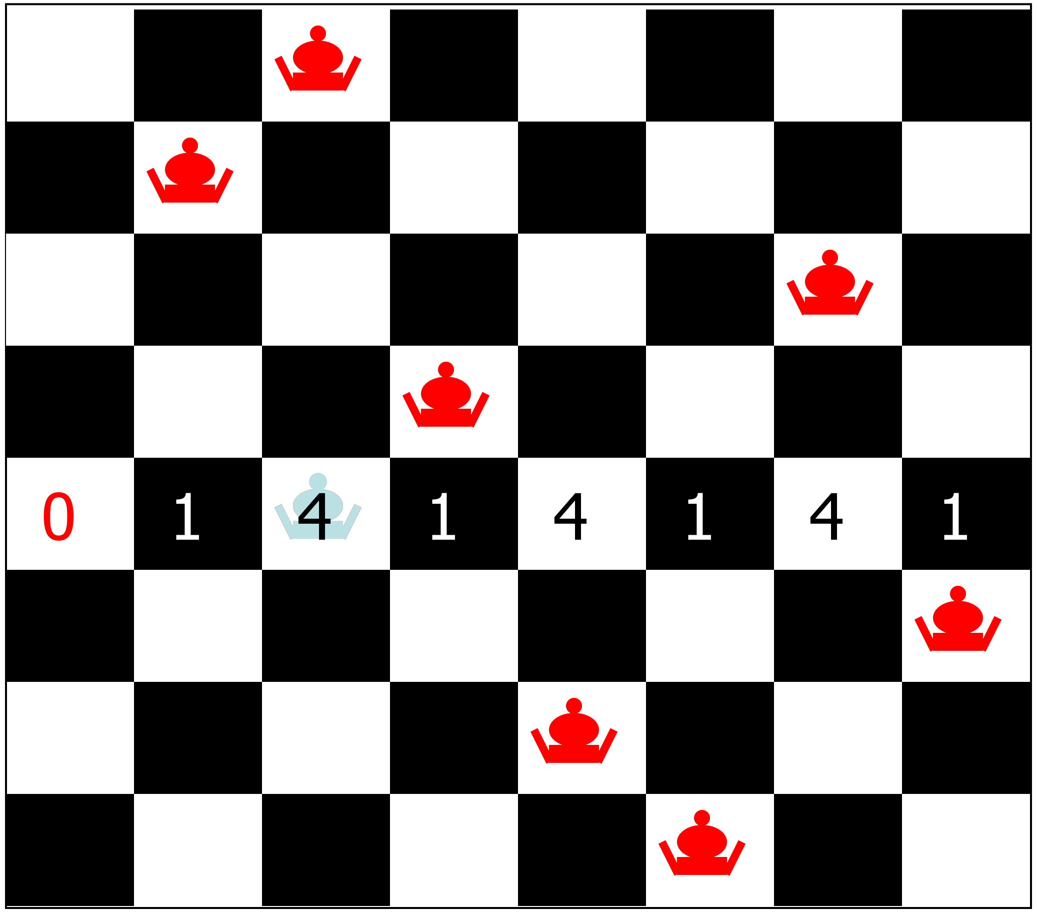
Eight Queens using Local Search

Take least cost move then try another Queen



Eight Queens using Local Search

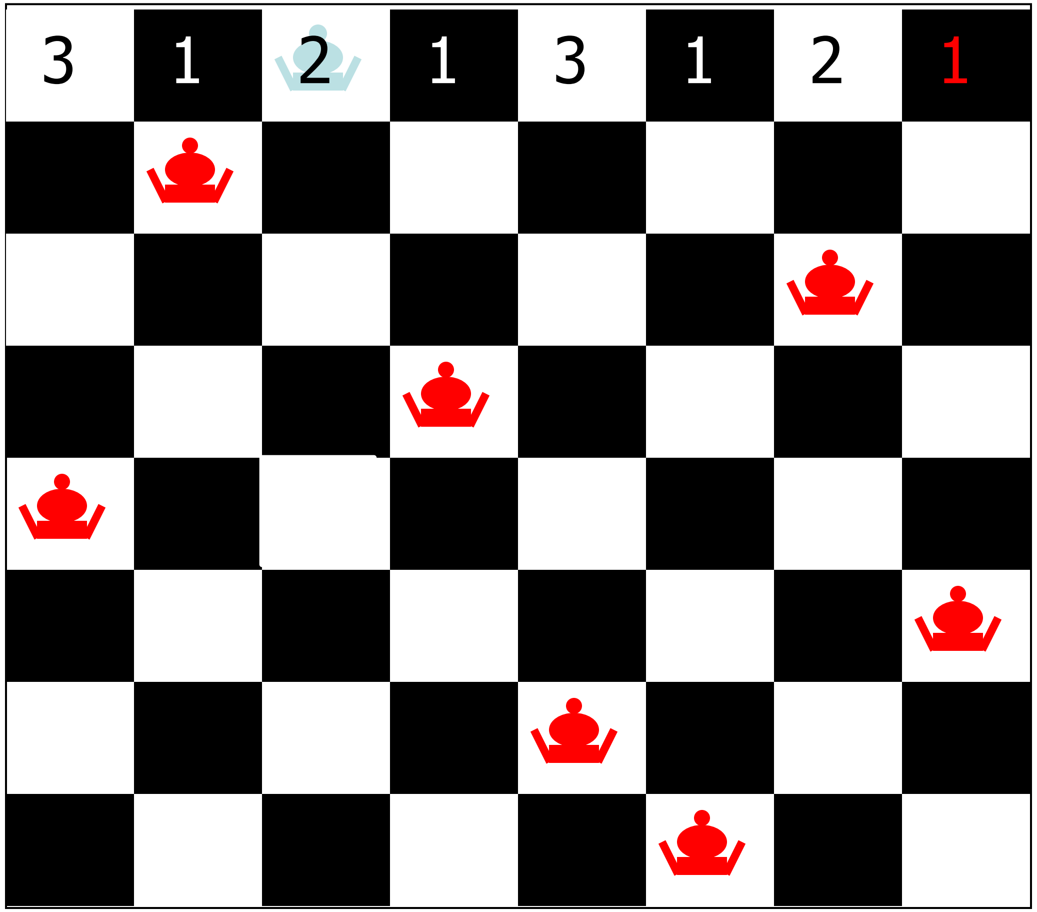
Take least cost move then try another Queen



Slide

Eight Queens using Local Search

Take least cost move then try another Queen

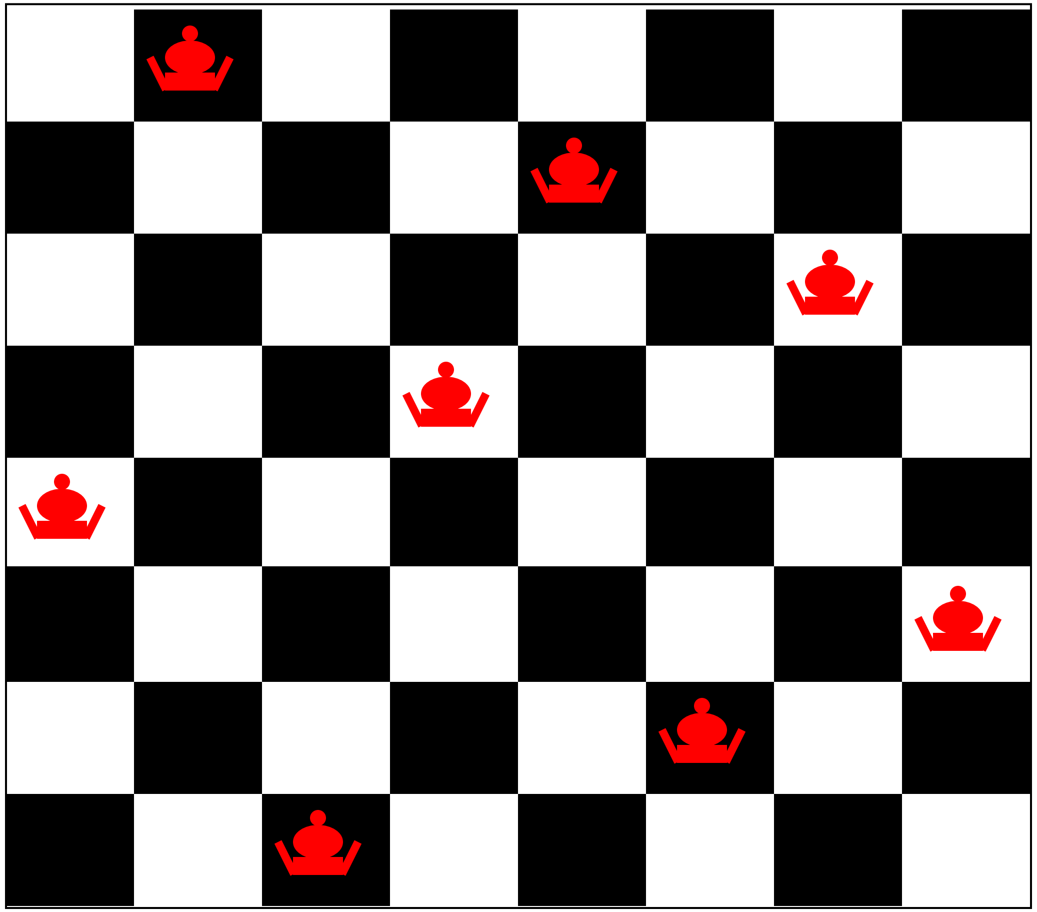


…and so on, until….

Slide

Eight Queens using Local Search

Answer Found



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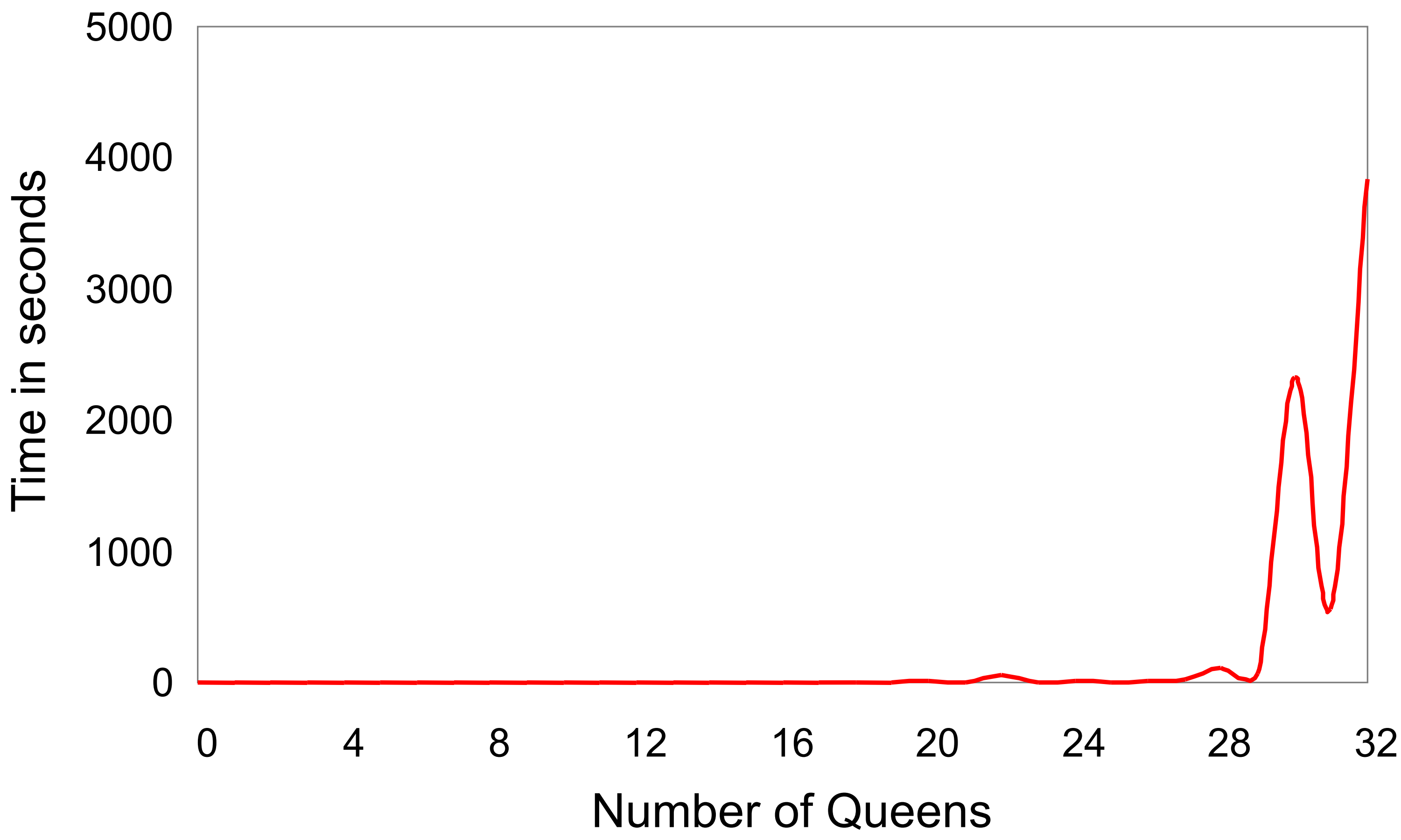
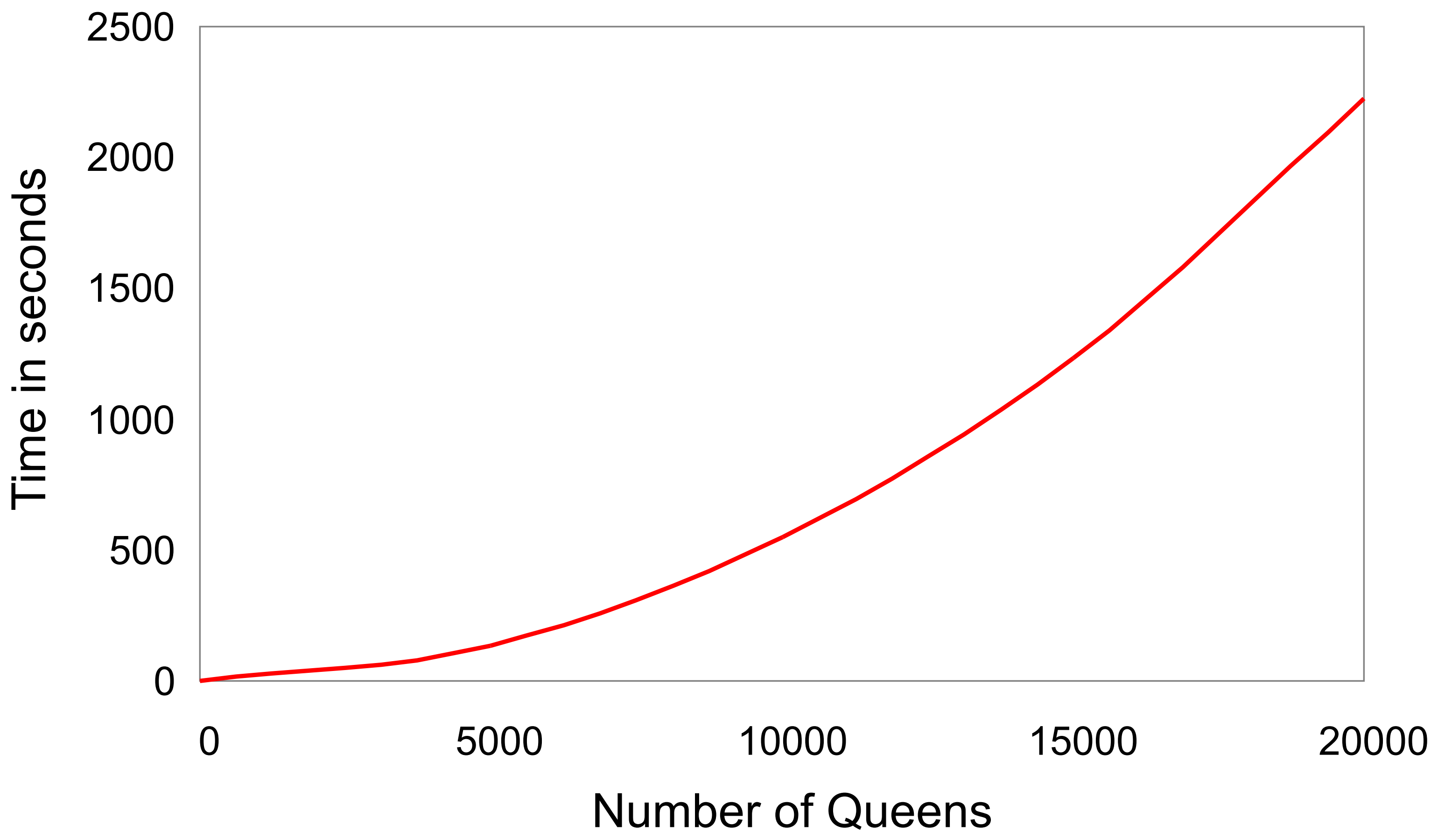
Video of Demo Iterative Improvement – Coloring

Backtracking Performance



Local Search Performance



Slide

Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)

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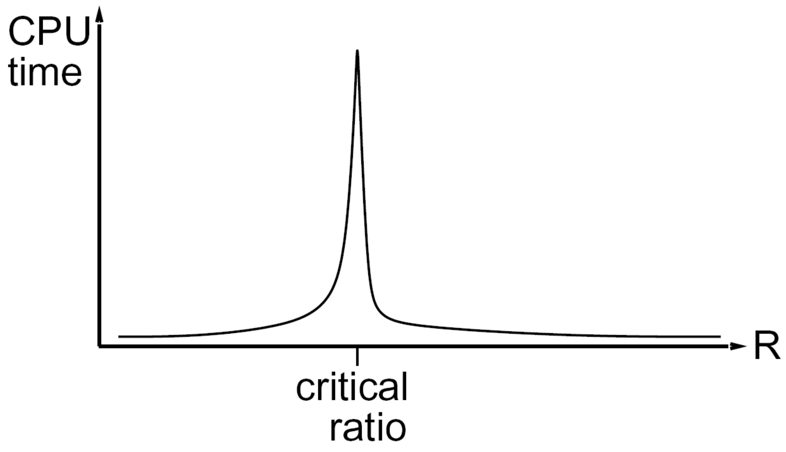

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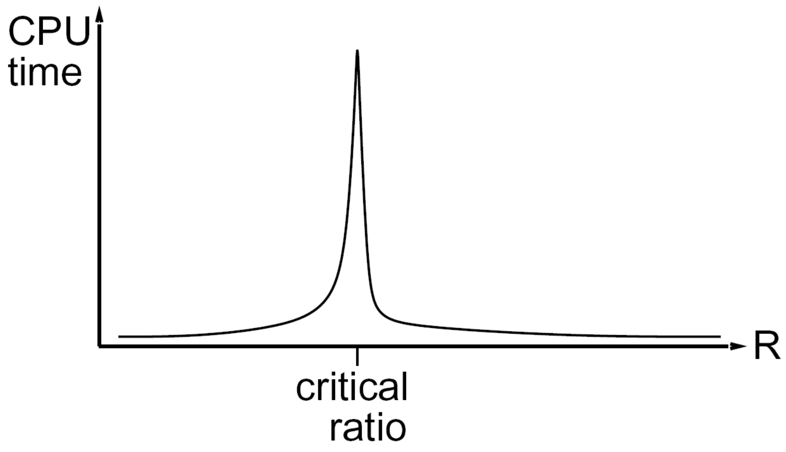

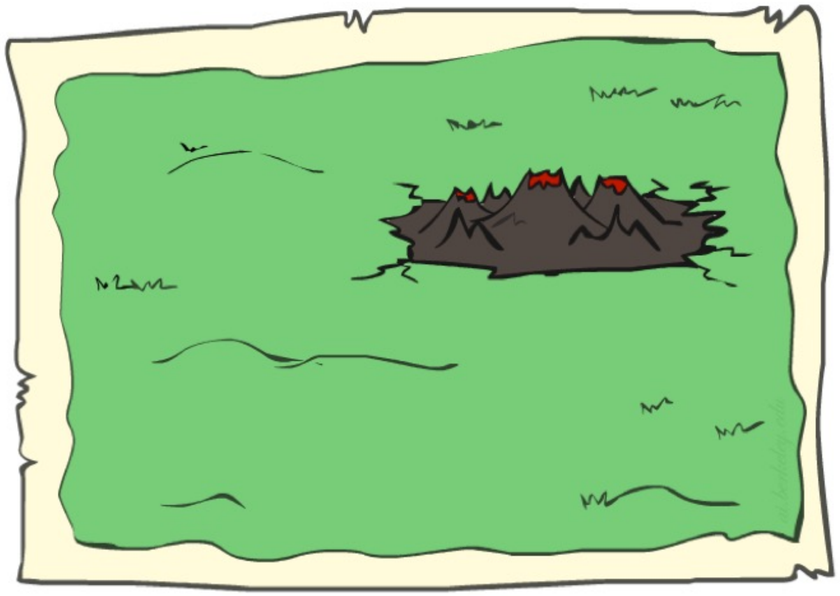

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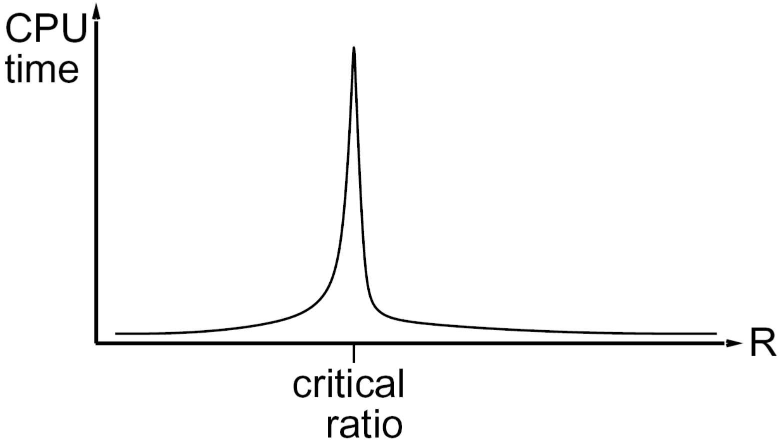

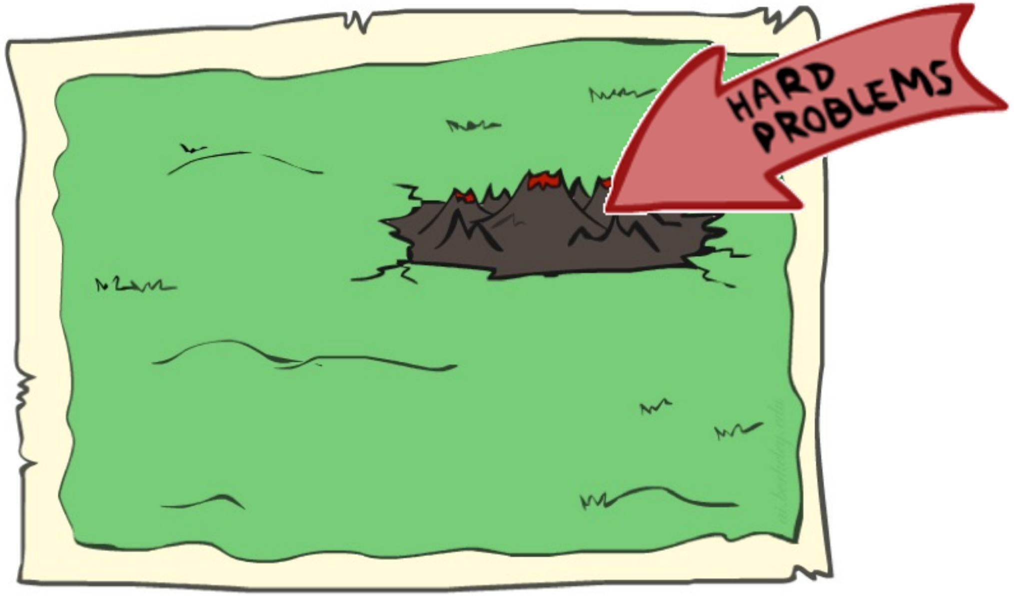

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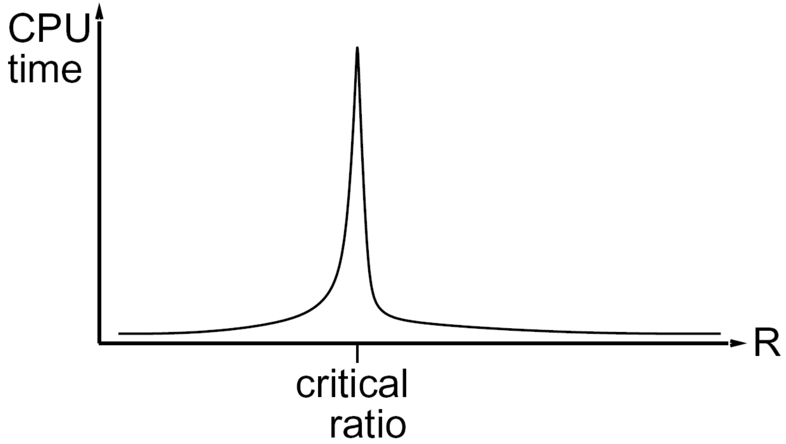

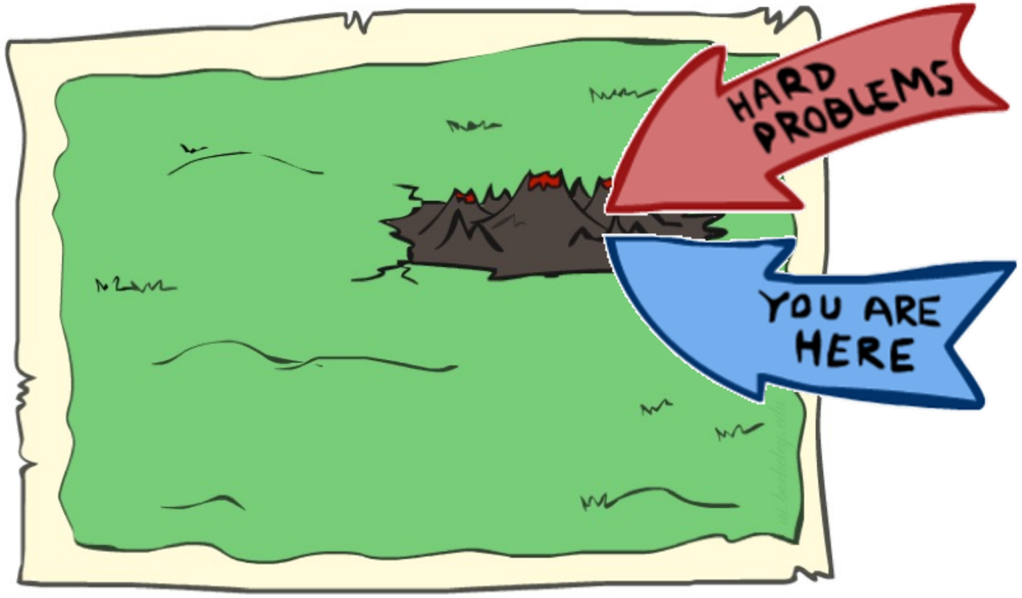

Summary: CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constrai
- Basic solution: backtracking sea
- Speed-ups:
  - Ordering
  - Filtering
  - Structure
- Iterative min-conflicts is often effective in practice