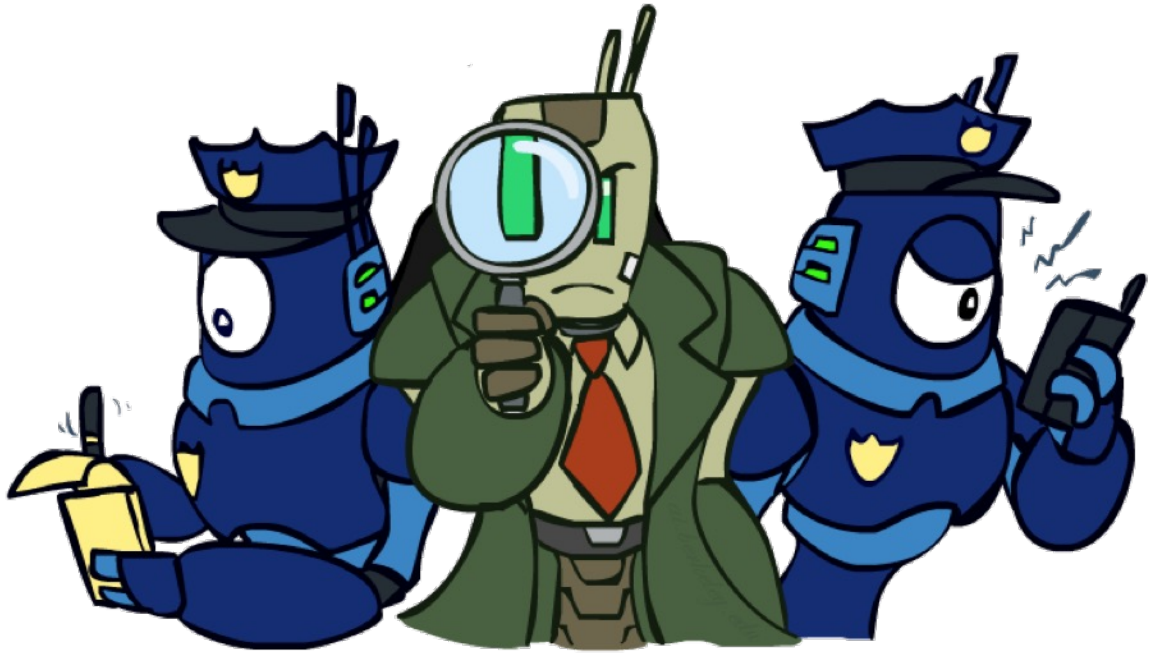
CMSC 471

Constraint Satisfaction Problems



Instructor: KMA Solaiman

These slides were created by Dan Klein and Pieter Abbeel at UC Berkeley. [ai.berkeley.edu]

What is Search For?

- Assumptions about the world: a single agent, deterministic actions, fully observed state, discrete state space

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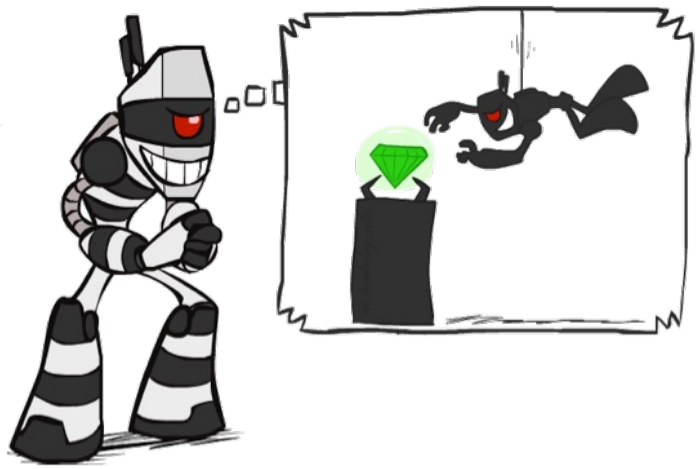

- Planning: sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance

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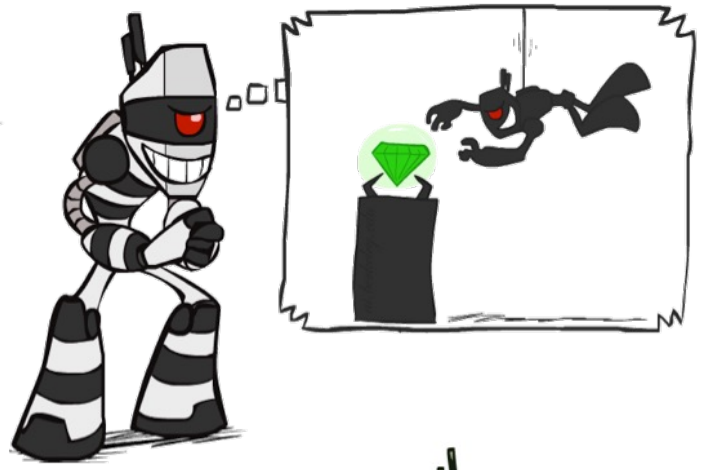
- Identification: assignments to variables
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are specialized for identification problems



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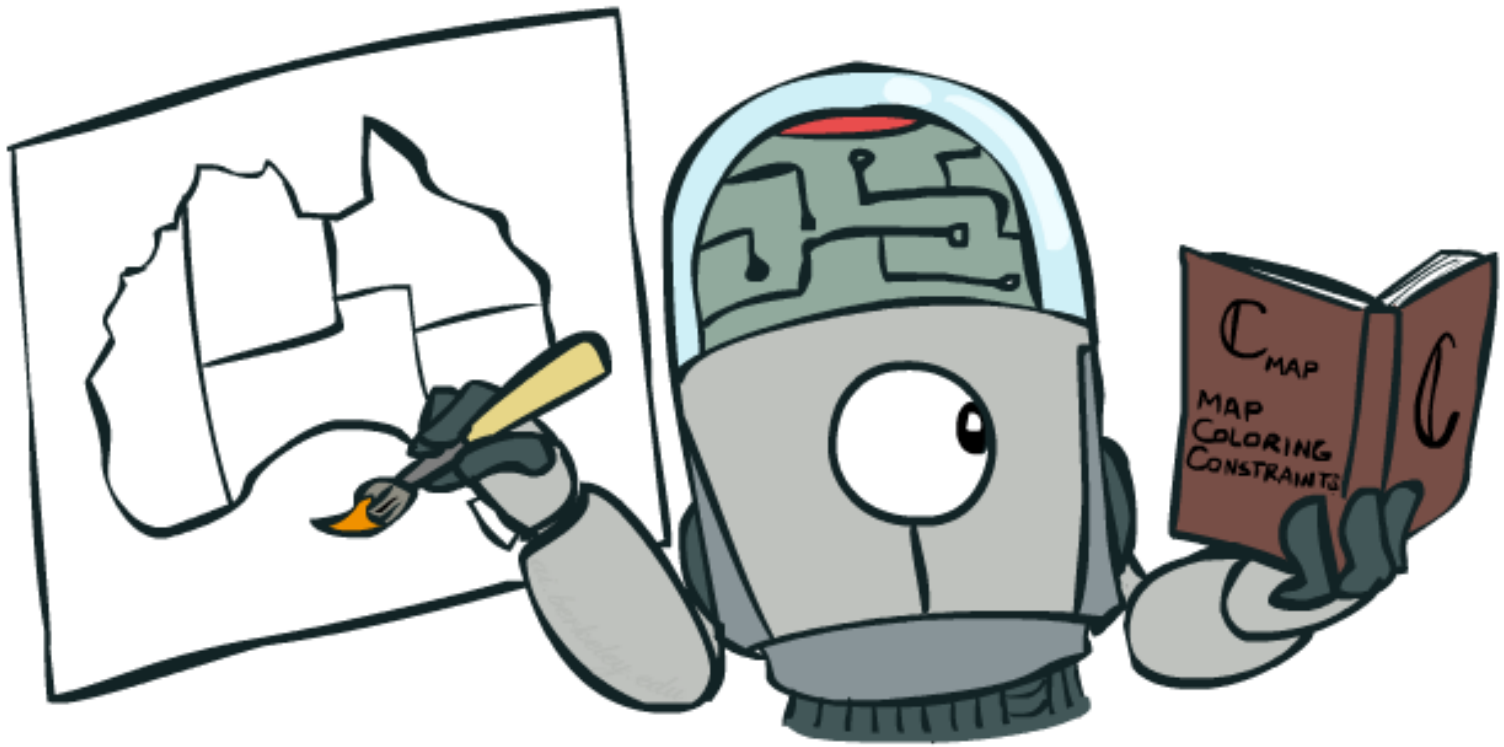

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- Standard search problems:
  - State is a “black box”: arbitrary data structure
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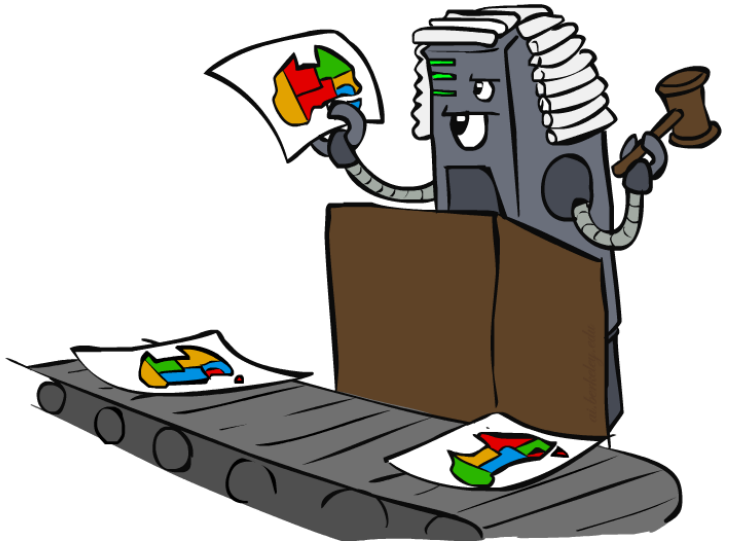

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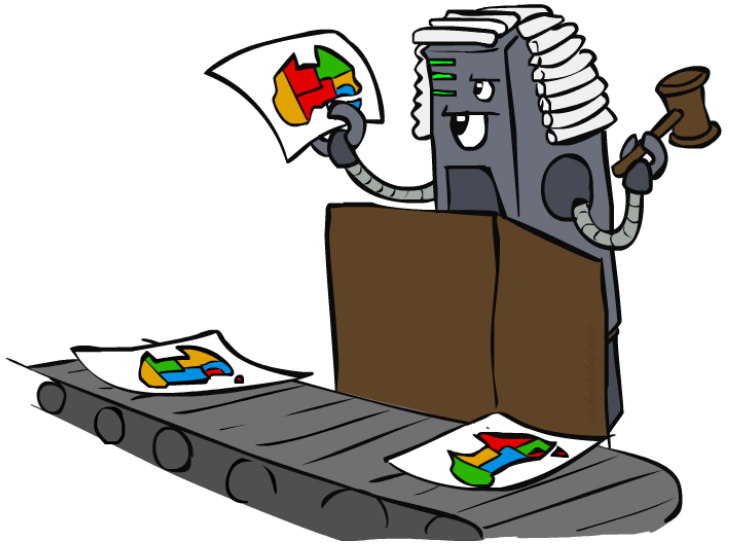
- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by variables X_i with values from a domain D (sometimes D depends on i)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables



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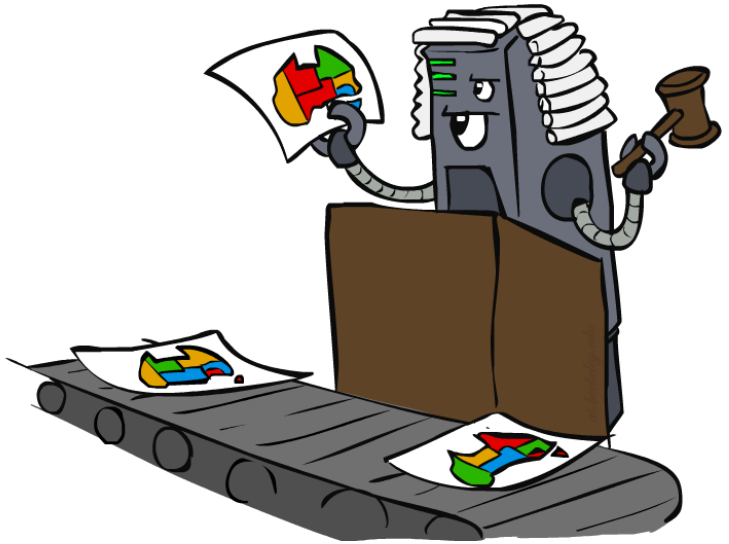

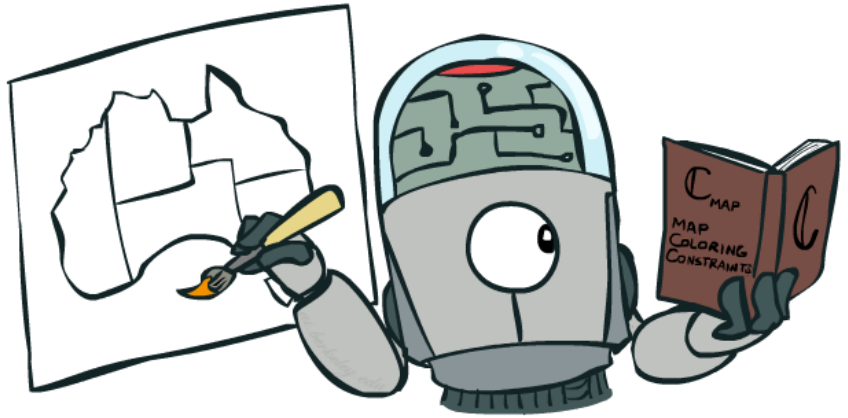

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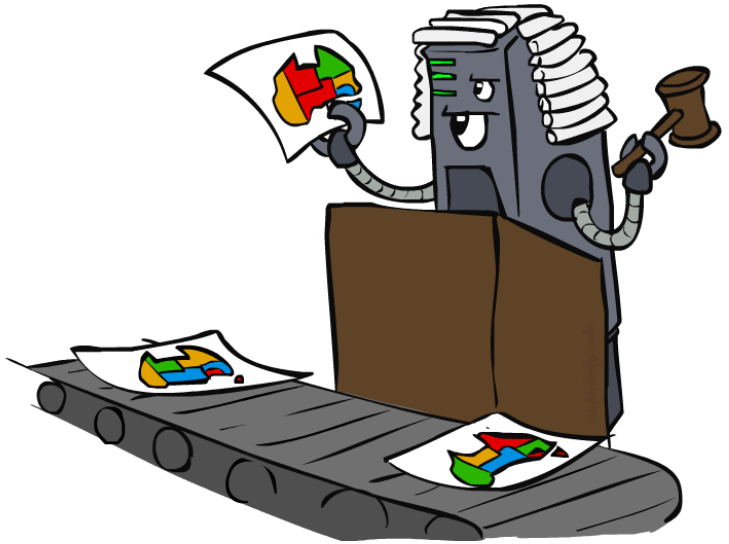
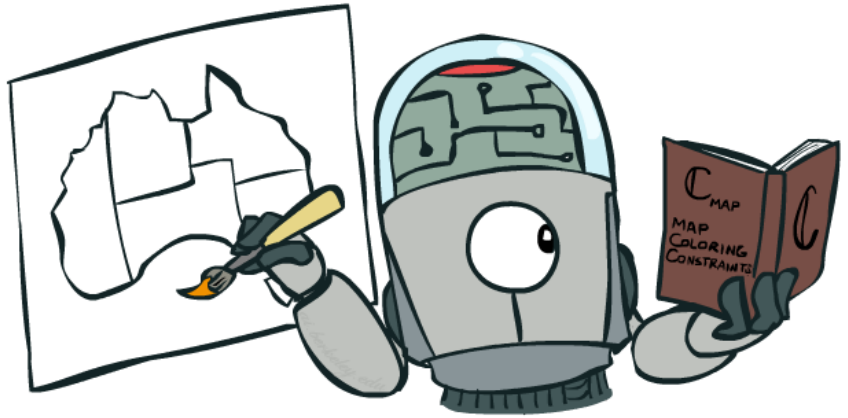
- Simple example of a *formal representation language*





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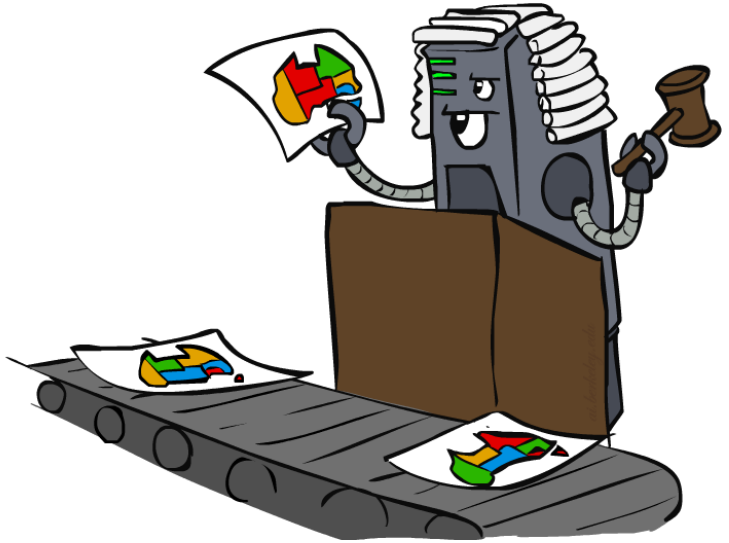
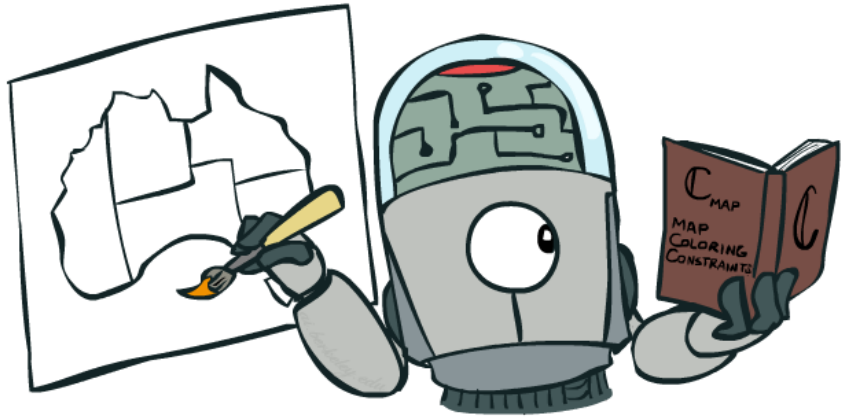
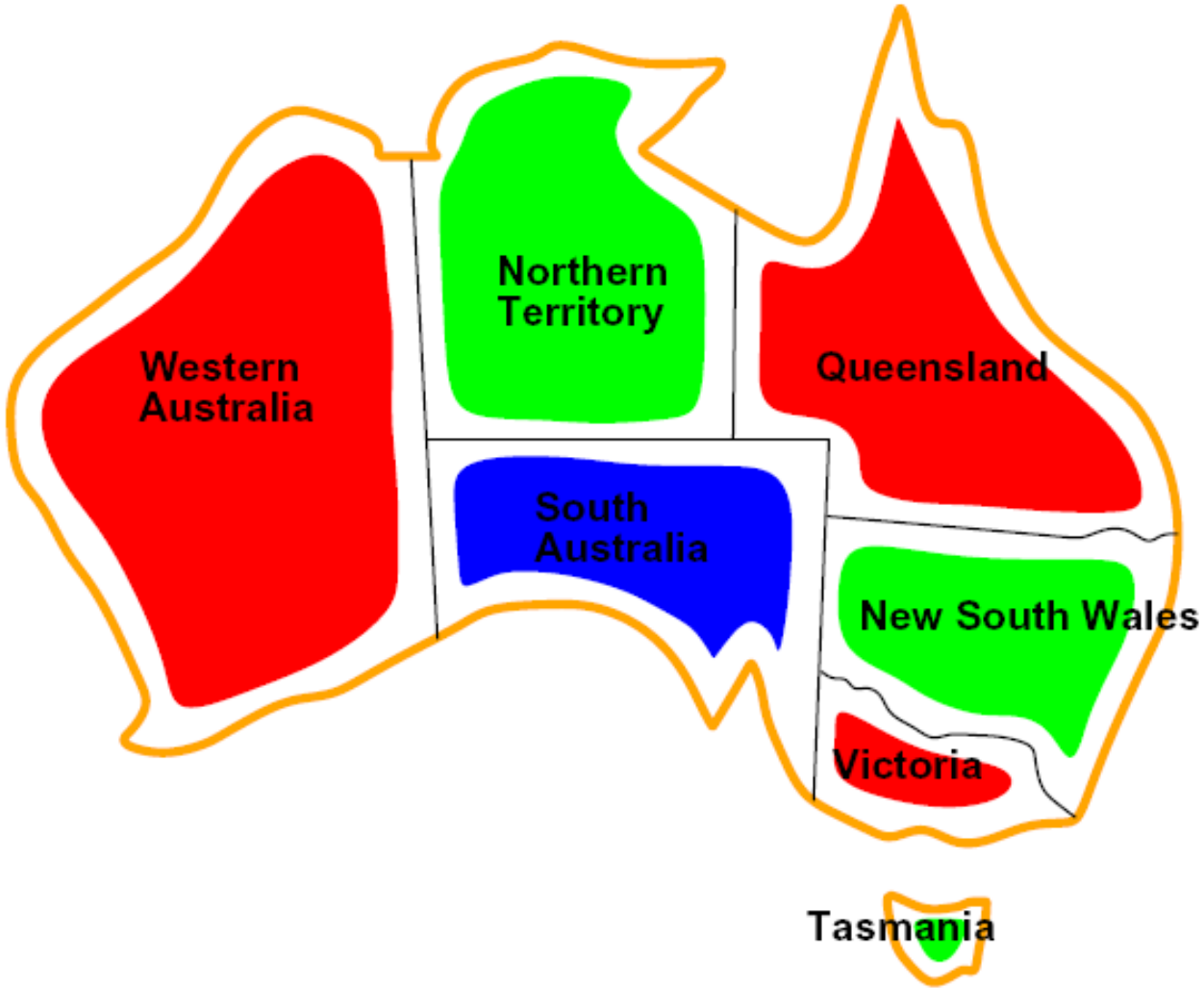
- Simple example of a *formal representation language*
- Allows useful general-purpose algorithms with more power than standard search algorithms





CSP Examples



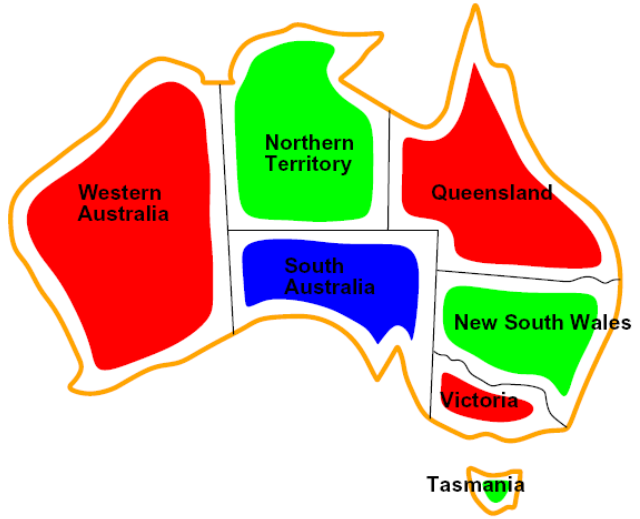
Example: Map Coloring

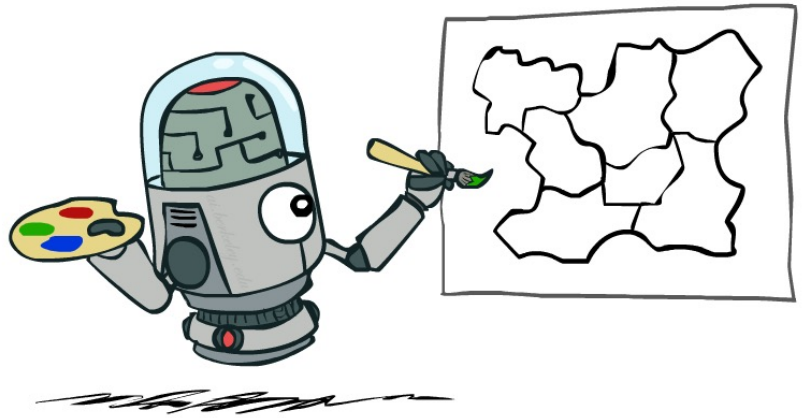
- Variables: $\mathrm{WA},\ \mathrm{NT},\ \mathrm{Q},\ \mathrm{NSW},\ \mathrm{V},\ \mathrm{SA},\ \mathrm{T}$
- Domains: $\mathrm{D} = \{\mathrm{red}, \mathrm{green}, \mathrm{blue}\}$
- Constraints: adjacent regions must have different colors

  Implicit: $\mathrm{WA} \neq \mathrm{NT}$

  Explicit: $(\mathrm{WA}, \mathrm{NT}) \in \{(\mathrm{red}, \mathrm{green}), (\mathrm{red}, \mathrm{blue}), \ldots\}$
- Solutions are assignments satisfying all constraints, e.g.:

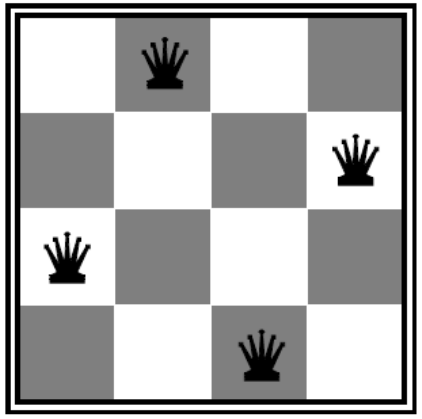
  $\{\mathrm{WA}{=}\mathrm{red},\ \mathrm{NT}{=}\mathrm{green},\ \mathrm{Q}{=}\mathrm{red},\ \mathrm{NSW}{=}\mathrm{green},$
  $\mathrm{V}{=}\mathrm{red},\ \mathrm{SA}{=}\mathrm{blue},\ \mathrm{T}{=}\mathrm{green}\}$

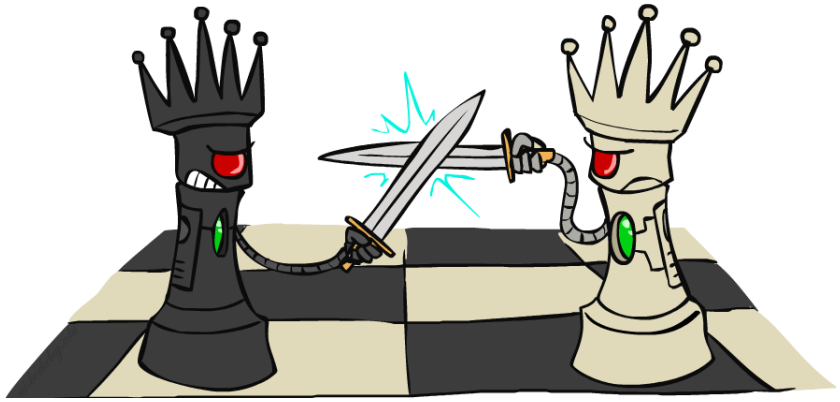




Example: N-Queens

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints



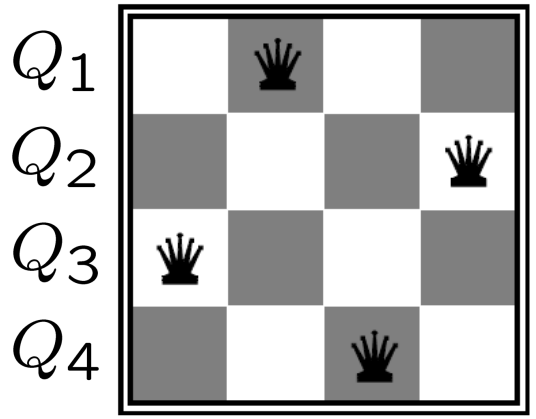


$$\forall i, j, k \quad (X_{ij}, X_{ik}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \quad (X_{ij}, X_{kj}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \quad (X_{ij}, X_{i+k,j+k}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \quad (X_{ij}, X_{i+k,j-k}) \in \{(0,0), (0,1), (1,0)\}$$

$$\sum_{i,j} X_{ij} = N$$

Example: N-Queens

- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, \ldots N\}$
  - Constraints:

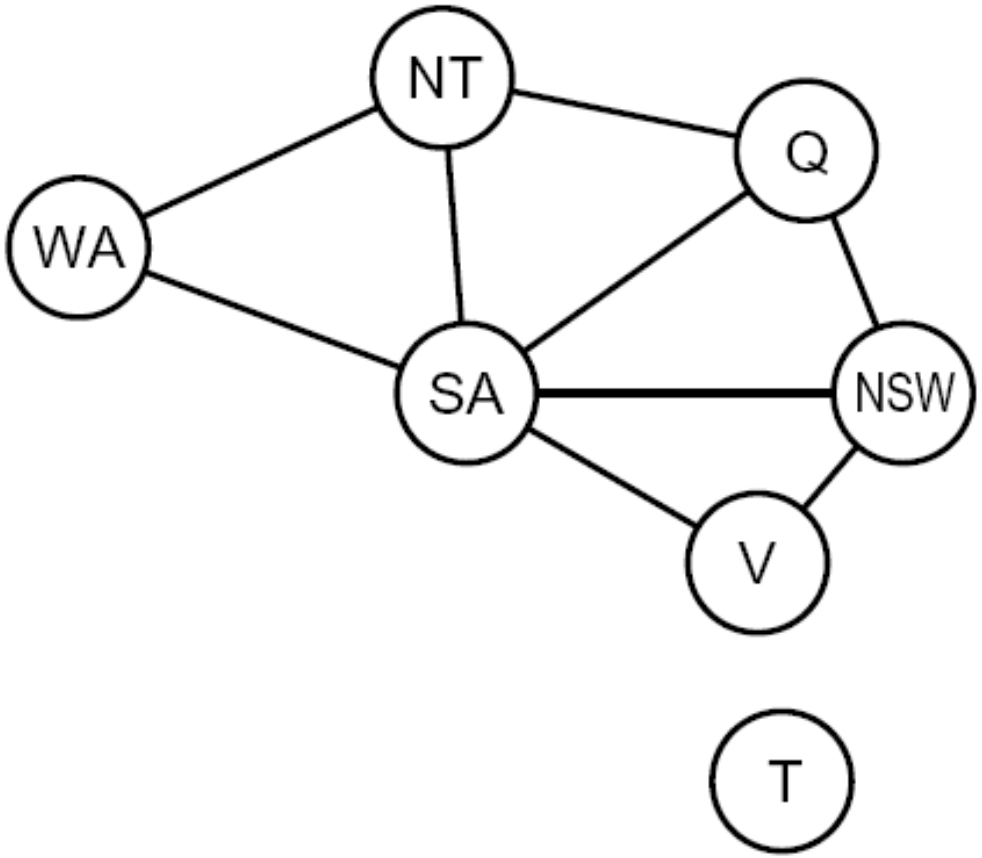


Implicit: $\forall i, j$ non-threatening(Q_i, Q_j)

Explicit: $(Q_1, Q_2) \in \{(1, 3), (1, 4), \ldots\}$

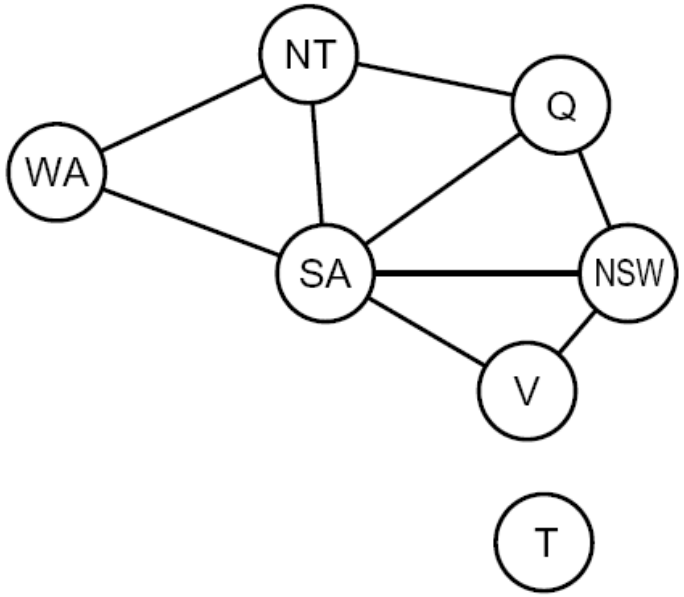
$\ldots$

Constraint Graphs



Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!



Example: Cryptarithmetic

- Variables:

  $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

- Domains:

  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

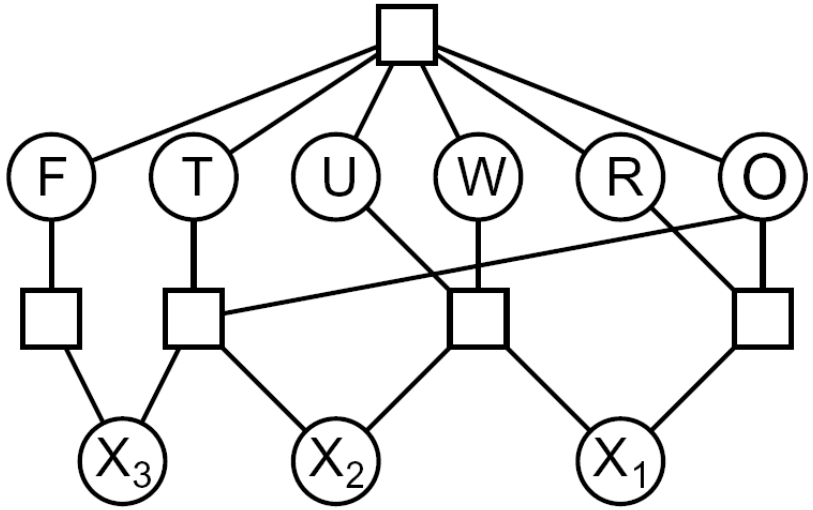
- Constraints:

  alldiff(F, T, U, W, R, O)

  $O + O = R + 10 \cdot X_1$

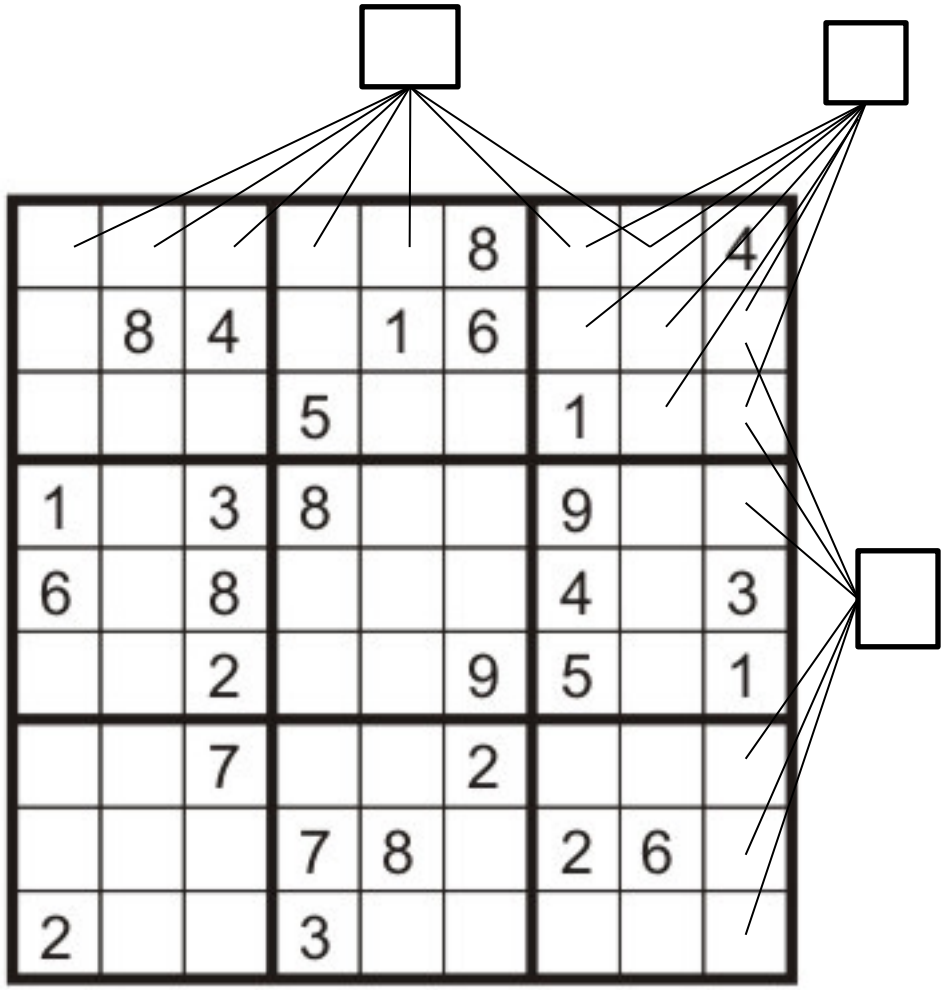
  $\cdots$

```
  T W O
+ T W O
-------
F O U R
```



0=7, R=4, W=6, U=2, T=8, F=1; 867 + 867 = 1734

Example: Sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

9-way alldiff for each column

9-way alldiff for each row

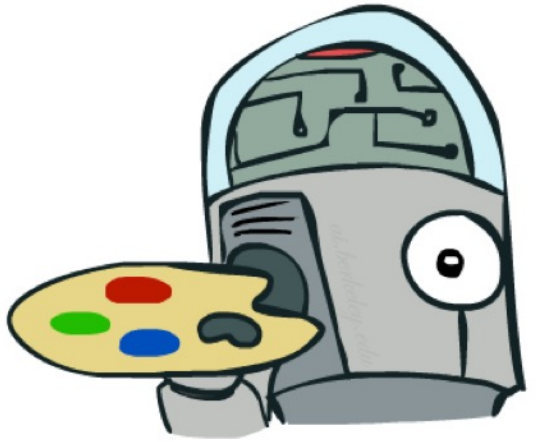
9-way alldiff for each region

(or can have a bunch of pairwise inequality constraints)

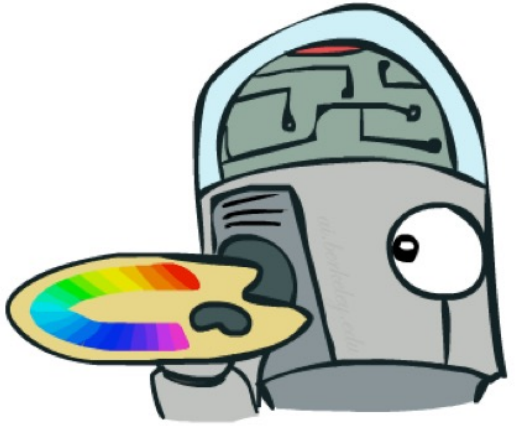
Varieties of CSPs and Constraints

Varieties of CSPs

- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable



- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time by LP methods



Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equivalent to reducing domains), e.g.:

    $$SA \neq green$$

  - Binary constraints involve pairs of variables, e.g.:

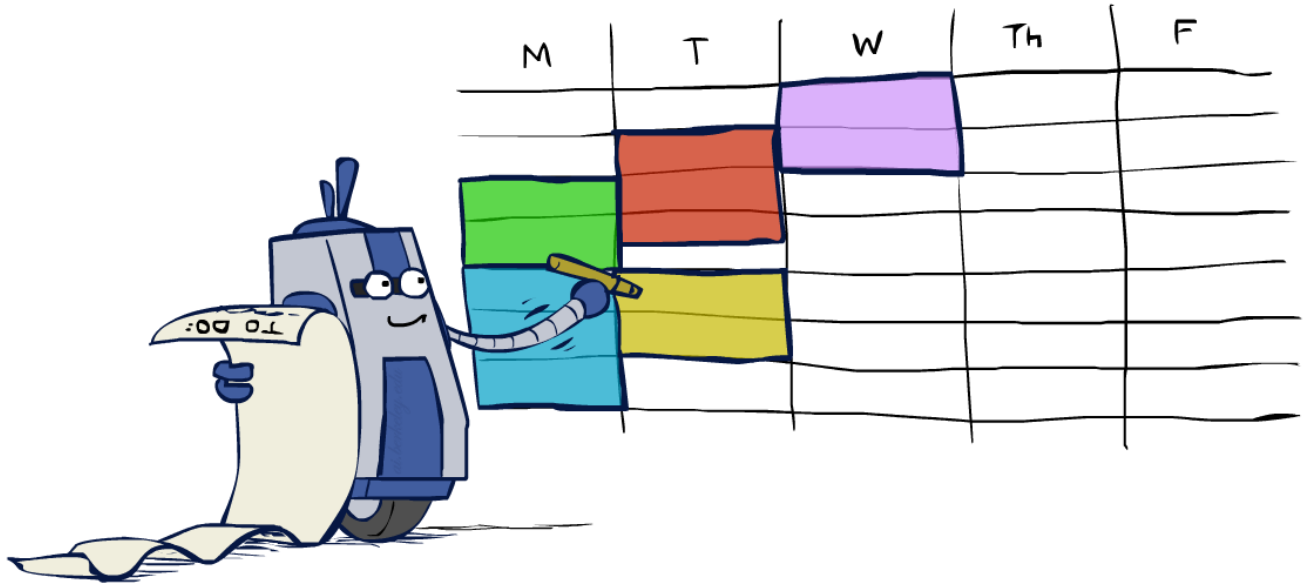
    $$SA \neq WA$$

  - Higher-order constraints involve 3 or more variables:
    e.g., cryptarithmetic column constraints

- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems
  - (We'll ignore these until we get to Bayes' nets)

Real-World CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Circuit layout
- Fault diagnosis
- … lots more!

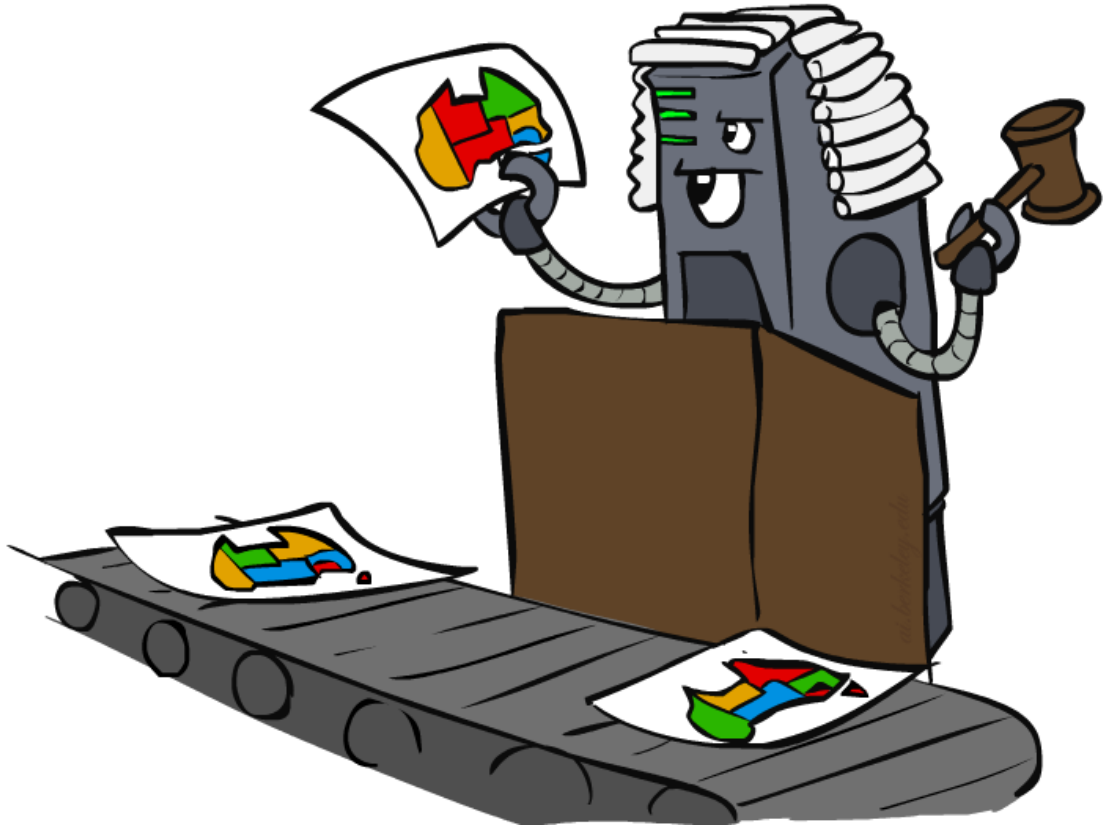


- Many real-world problems involve real-valued variables…

Solving CSPs

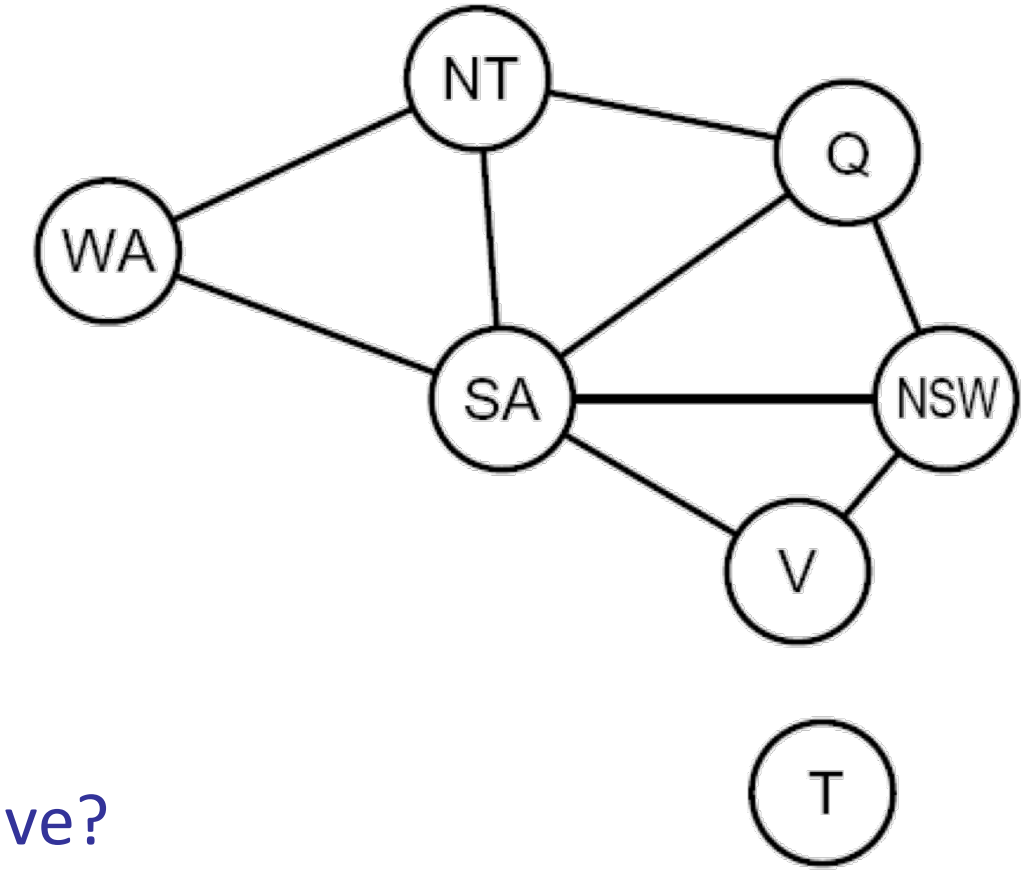
Standard Search Formulation

- Standard search formulation of CSPs
- States defined by the values assigned so far (partial assignments)
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints
- We'll start with the straightforward, naïve approach, then improve it



Search Methods

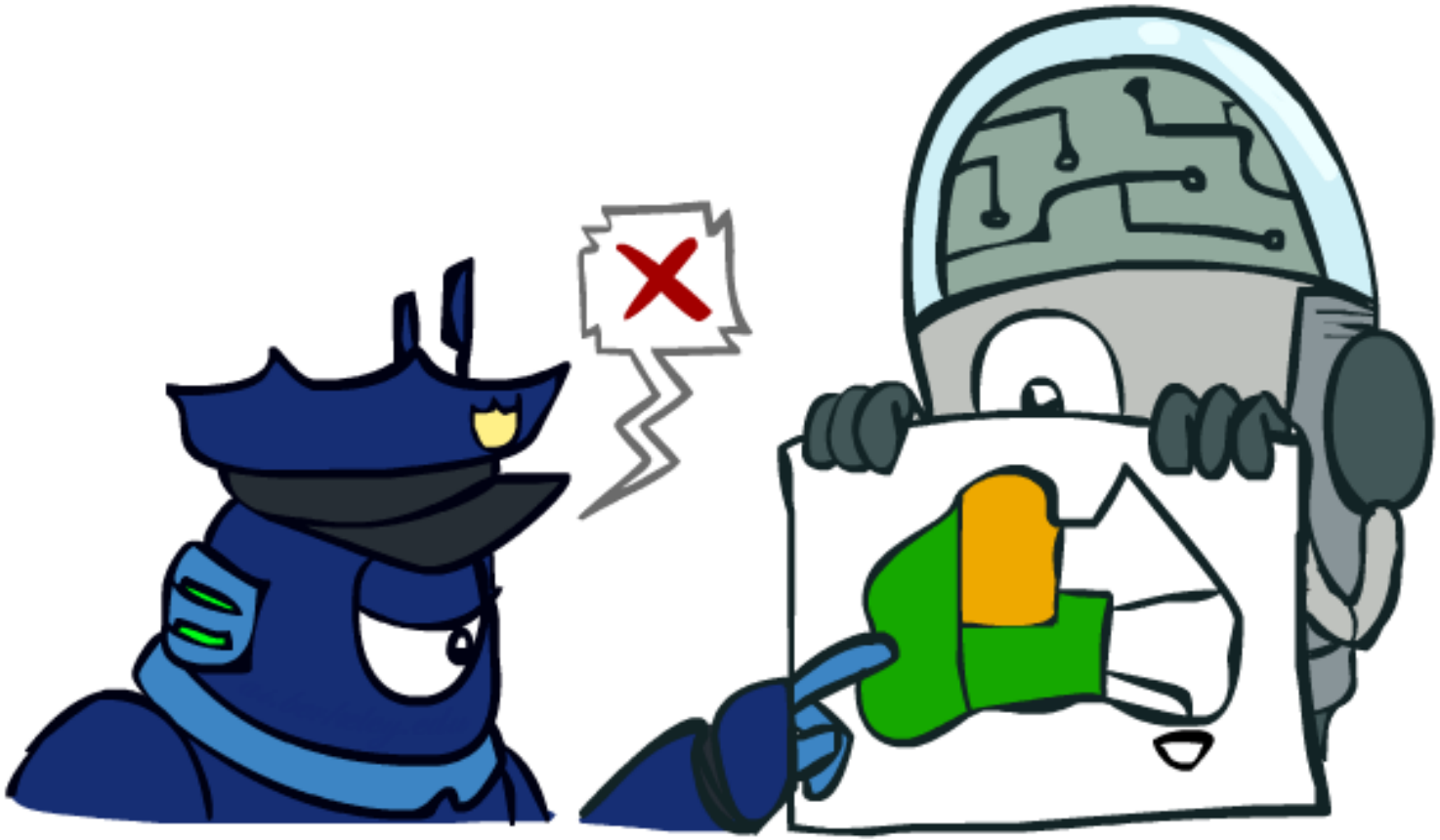
- What would BFS do?
- What would DFS do?
- What problems does naïve search have?



[Demo: coloring -- dfs]

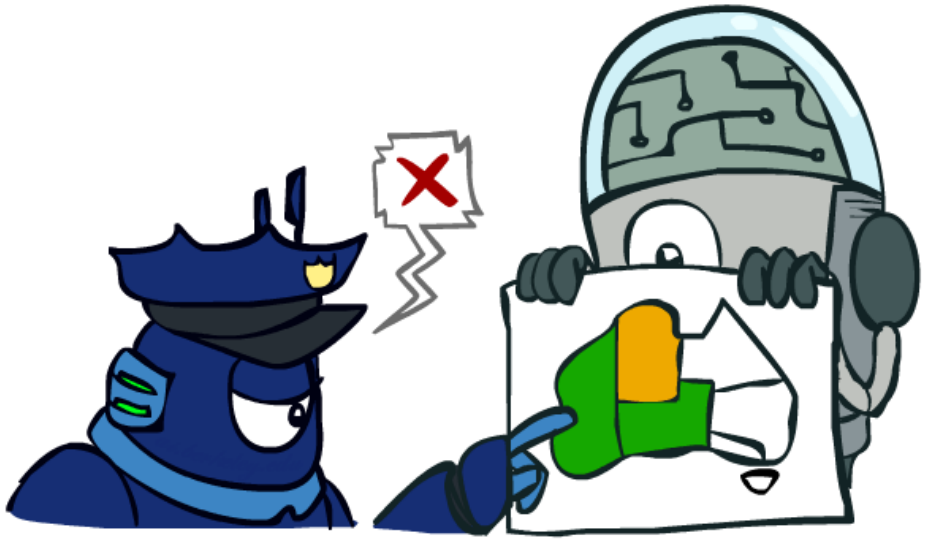
Video of Demo Coloring -- DFS

Backtracking Search

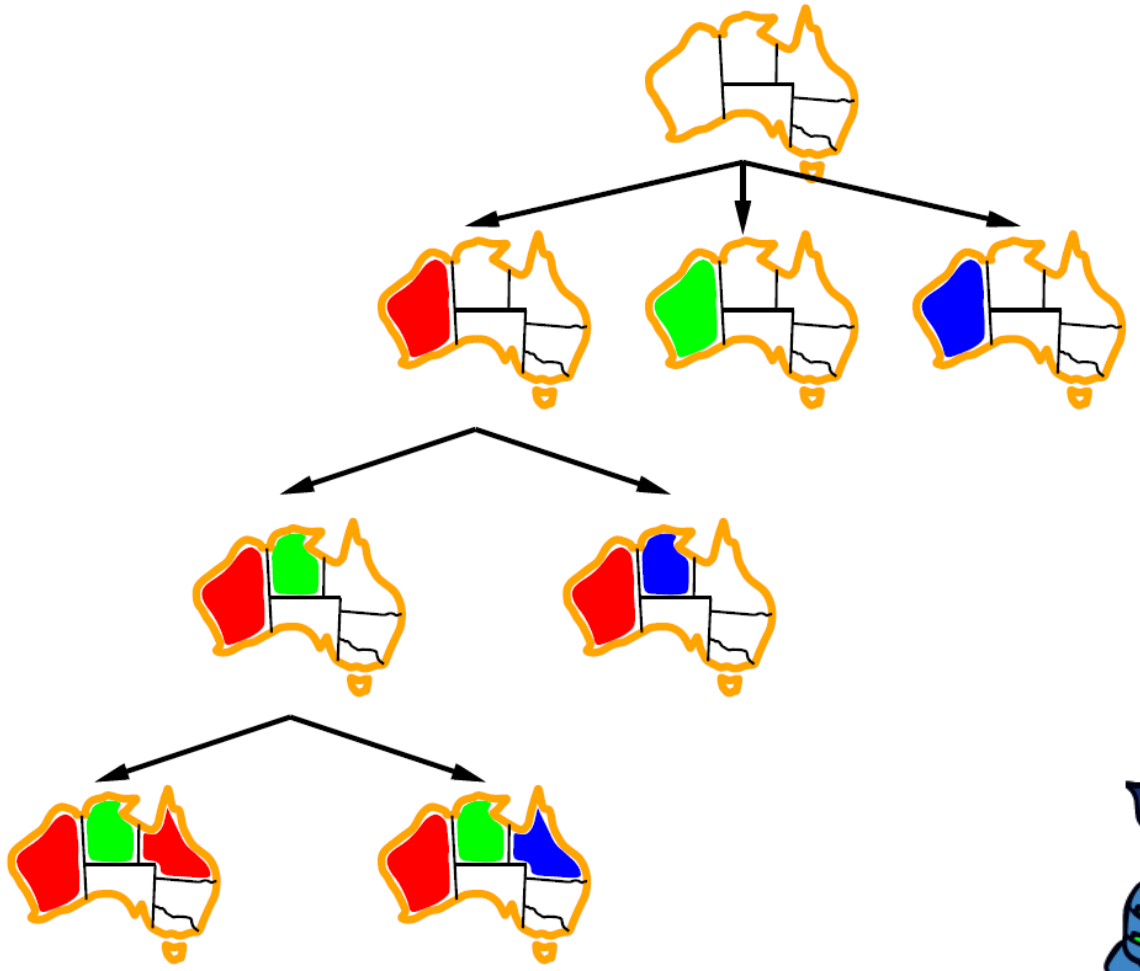


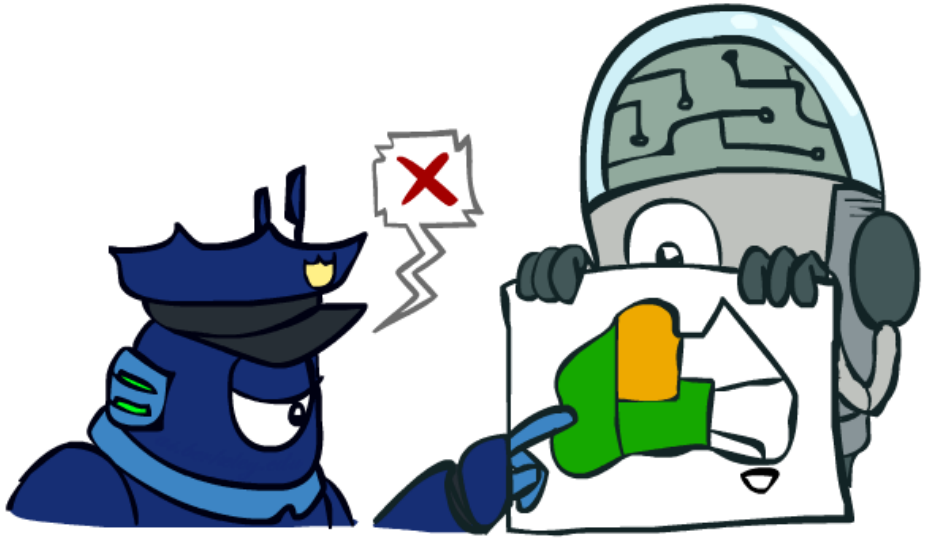
Backtracking Search

- Backtracking search is the basic uninformed algorithm for solving CSPs
- Idea 1: One variable at a time
  - Variable assignments are commutative, so fix ordering
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
- Idea 2: Check constraints as you go
  - I.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to check the constraint
  - "Incremental goal test"
- Depth-first search with these two improvements is called *backtracking search* (not the best name)
- Can solve n-queens for n ≈ 25



Backtracking Example





Video of Demo Coloring – Backtracking

Backtracking Search

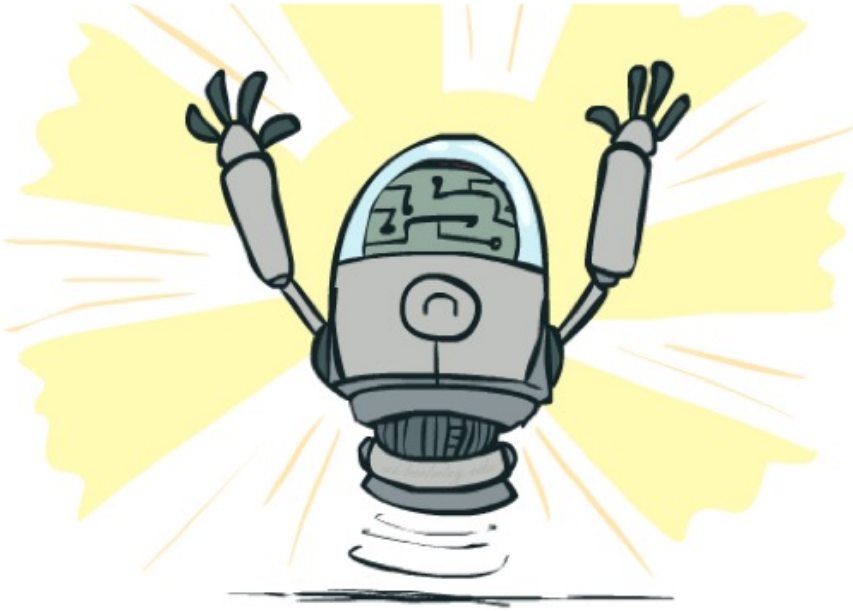
```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

- Backtracking = DFS + variable-ordering + fail-on-violation
- What are the choice points?

[Demo: coloring -- backtracking]

Improving Backtracking

- General-purpose ideas give huge gains in speed
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: Can we detect inevitable failure early?
- Structure: Can we exploit the problem structure?

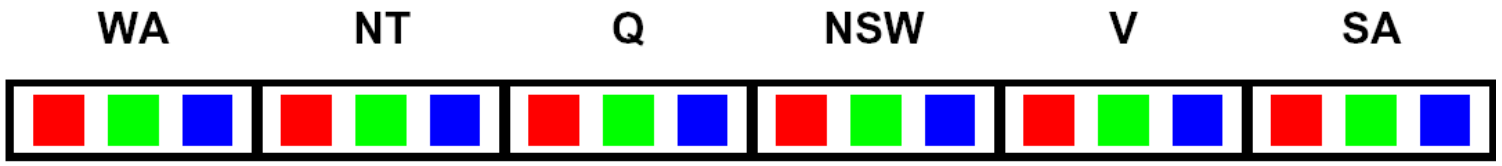


Filtering

Keeping track of domains for unassigned variables and cross off bad options

Filtering: Forward Checking

- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: Cross off values that violate a constraint when added to the existing assignment



[Demo: coloring -- forward checking]

Video of Demo Coloring – Backtracking with Forward Checking